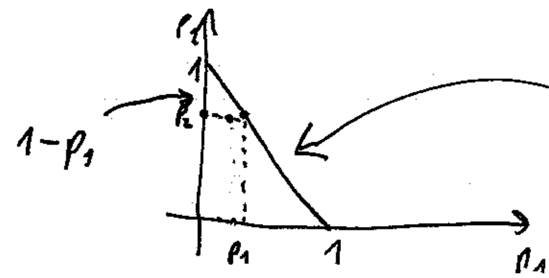
2.1 LOTTERIES

LOTTERY L IS A SET OF OUTCOMES WITH ASSOCIATED PROBABILITIES

$$L = (x_1, p_1;\ x_2, p_2;\ x_3, p_3;\ \dots;\ x_S, p_S) \qquad \sum_{i=1}^{S} p_i = 1 \qquad p_i \in [0,1]$$

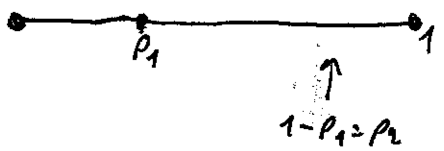
ONLY $p_1 \cdots p_{S-1}$ ARE INDEPENDENT SINCE $p_S = 1 - \sum_{i=1}^{S-1} p_i$

- IF $S=2$ WE CAN REPRESENT A LOTTERY

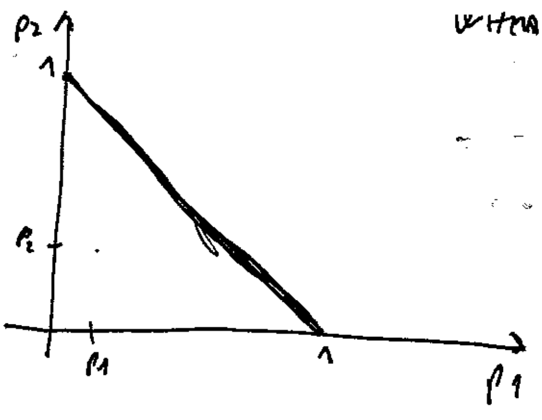


WITH A POINT IN $\mathbb{R}^2$ SPACE. THIS POINT CLEARLY HAS COORDINATES BETWEEN 0 AND 1. MOREOVER $p_1+p_2=1$ AND THEREFORE $p_2 = 1-p_1$ MEANING THAT THE POINT STAYS ON THE $y=1-x$ LINE

WE CAN REPRESENT THE SAME POINT ON A SEGMENT USING ONLY p_1, SINCE p_2 IS NOT INDEPENDENT



- IF $S=3$ WE CAN REPRESENT THE TWO INDEPENDENT p_1 AND p_2



WHERE $p_3 = 1-p_1-p_2$. THESE VALUES STAY BETWEEN 0 AND 1.

MACHINA TRIANGLE M

2.2

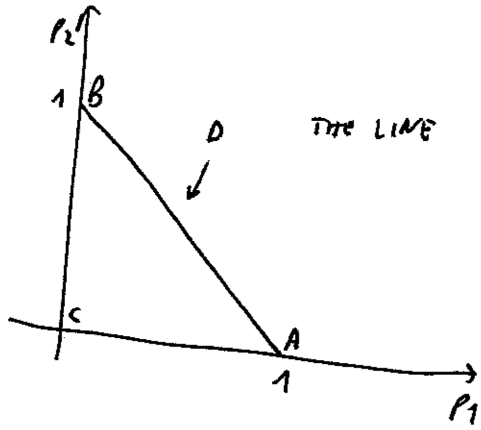
NOTE THAT Ⓐ IF $p_1 = 1 \Rightarrow p_2 = 0 \; p_3 = 0$

Ⓑ IF $p_2 = 1 \Rightarrow p_1 = 0 \; p_3 = 0$

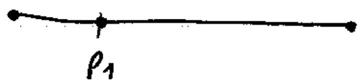
Ⓒ IF $p_1 = 0$ AND $p_2 = 0 \Rightarrow p_3 = 1$

Ⓓ IF $p_3 = 0 \Rightarrow p_1 + p_2 = 1$

THESE POINTS ARE



NOTE THAT M_2 IS THIS

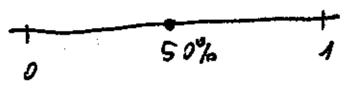


(FOR $S = 2$)

<u>DEFINITION</u>: IF $p_i = 1$, THE LOTTERY IS DEGENERATED, OR DETERMINISTIC. ITS ONLY POSSIBLE OUTCOME IS x_i

<u>EXERCISES</u>: • DESCRIBE AND DRAW MACHINA TRIANGLE FOR LOTTERY COIN FLIPPING

$L = (\text{HEAD}, 50\%; \text{TAIL}, 50\%)$ $M_2 =$



• DIE ROLL

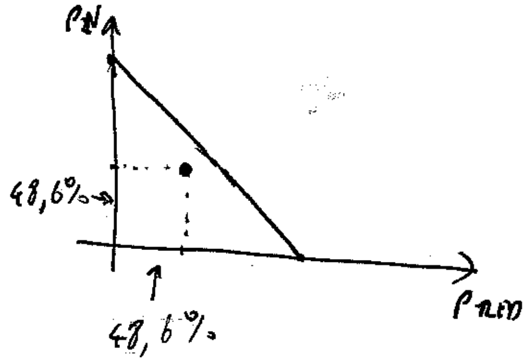
$L = (1, 1/6; 2, 1/6; 3, 1/6; 4, 1/6; 5, 1/6; 6, 1/6)$ M_5 IS A 5-DIMENSIONAL OBJECT AND IT IS IMPOSSIBLE TO DRAW

2.3)

- ROUGE/NOIR ROULETTE

$L = \left(\text{RED}, 18/37 \; ; \text{BLACK}, 18/37 \; ; \text{ZERO}, 1/37\right) \approx \left(R, 48{,}6\% \; ; B, 48{,}6\% \; ; 0, 2{,}7\%\right)$

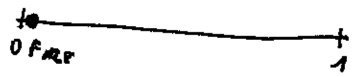
M_3



THE POINT IS CLOSE TO THE LINE, MEANING THAT $P_3 \approx 0$

- HOUSE FIRE (YEARLY)

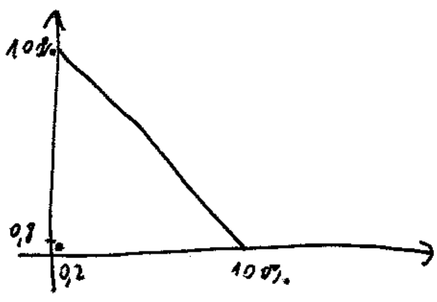
$L = \left(\text{FIRE}, 1\% \; ; \text{NOTHING}, 99\%\right)$ M_2



- HOUSE FIRE DAMAGE (YEARLY)

$L = \left(-100'000\text{ €}, 0{,}2\% \; ; -20'000\text{ €}, 0{,}8\% \; ; 0, 99\%\right)$

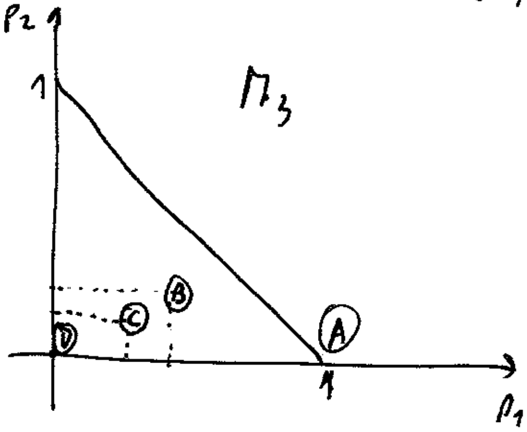
$M_3 =$



IT IS VERY FAR FROM THE LINE, MEANING THAT $P_3 \approx 100\%$

2.4

- IN M_3, SHOW $A=(1,0,0)$ $B=(1/2, 1/3, 1/6)$ $C=(0,3;\ 0,2;\ 0,5)$ $D=(0,0,1)$



SOME CONSIDERATIONS ON LOTTERIES

- GIVEN TWO LOTTERIES WITH DIFFERENT OUTCOMES,

$$L_1 = (x_1, p_1;\ x_2, p_2;\ \dots;\ x_s, p_s) \qquad L_2 = (y_1, q_1;\ y_2, q_2;\ \dots;\ y_k, p_k)$$

WE CAN ALWAYS "MODIFY" THEM TO HAVE THE SAME OUTCOMES

$$L_1 = (x_1, p_1; x_2, p_2; \dots; x_s, p_s;\ y_1, 0;\ y_2, 0;\ \dots;\ y_k, 0)$$

$$L_2 = (x_1, 0;\ x_2, 0;\ \dots;\ x_s, 0;\ y_1, q_1;\ y_2, q_2;\ \dots\ y_k, q_k)$$

SO FROM NOW ON, EVERYTIME WE CONSIDER LOTTERIES, WE MAY SUPPOSE THAT THEY HAVE THE SAME OUTCOMES.

- FROM NOW ON, OUTCOMES x_i WILL BE NUMBERS, AND VERY OFTEN MONEY!

- LOTTERIES CAN BE COMPOSED IN THIS WAY $L^A = (x_1, p_1^A; \dots x_s, p_s^A)$

$$L^B = (x_1, p_1^B; \dots; x_s, p_s^B) \qquad \alpha L^A + (1-\alpha) L^B = (x_1, \alpha p_1^A + (1-\alpha) p_1^B;\ \dots;\ x_s, \alpha p_s^A + (1-\alpha) p_s^B)$$

NOTE THAT THE SUM OF COMPOSITION'S COEFFICIENTS MUST BE 1

2.5

THE SUM OF COEFFICIENTS OF A LOTTERIES' COMPOSITION MUST BE ONE BECAUSE:

$$\alpha L^A + \beta L^B = \left(x_1, \alpha p_1^A + \beta p_1^B ; \dots ; x_S, \alpha p_S^A + \beta p_S^B \right)$$

THIS IS A LOTTERY ONLY IF $\sum_{i=1}^{S} \left(\alpha p_i^A + \beta p_i^B \right) = 1$

BUT WE ALREADY KNOW THAT $\sum_{i=1}^{S} p_i^A = 1$ $\sum_{i=1}^{S} p_i^B = 1$

AND THEREFORE $\sum_{i=1}^{S} \alpha p_i^A + \sum_{i=1}^{S} \beta p_i^B = \alpha \sum_{i=1}^{S} p_i^A + \beta \sum_{i=1}^{S} p_i^B = \alpha \cdot 1 + \beta \cdot 1$

AND $\alpha + \beta$ MUST BE EQUAL TO ONE.

LOOK AT A LOTTERY AND LOOK AT A RANDOM VARIABLE
THEY ARE THE SAME.

THEY BOTH HAVE OUTCOMES, PROBABILITIES AND $\sum_{i=1}^{S} p_i = 1$

THEREFORE WE CAN DEFINE ALSO CONTINOUS LOTTERIES

L WITH OUTCOMES ON $\mathbb{R}$ (OR A SUBSET OF $\mathbb{R}$)

AND A DENSITY OF PROBABILIT[Y] $f_L(x)$ SUCH THAT

$$P(x \in [a,b]) = \int_a^b f_L(s)\, ds$$

2.5 BIS

EXERCISE

CONSIDER $L^1 = (1, 1/3;\ 2, 1/3;\ 4, 1/3)$ AND

$L^2 = (2, 1/2;\ 3, 1/4;\ 5, 1/4)$

CALCULATE THE COMPOSITION OF L^1 AND L^2 WITH $\alpha = 0.2$

FIRST WE MUST FIX THE TWO LOTTERIES SO THAT THEY HAVE THE SAME OUTCOMES

$$L^1 = (1, 1/3;\ 2, 1/3;\ 4, 1/3;\ 3, 0;\ 5, 0)$$

$$L^2 = (1, 0;\ 2, 1/2;\ 4, 0;\ 3, 1/4;\ 5, 1/4)$$

AND NOW

$$\alpha L^1 + (1-\alpha) L^2 = \Big(1,\ 0.2 \cdot \tfrac{1}{3} + 0.8 \cdot 0;\ 2,\ 0.2 \cdot \tfrac{1}{3} + 0.8 \cdot \tfrac{1}{2};$$

$$4,\ 0.2 \cdot \tfrac{1}{3} + 0.8 \cdot 0;\ 3,\ 0.2 \cdot 0 + 0.8 \cdot \tfrac{1}{4};\ 5,\ 0.2 \cdot 0 + 0.8 \cdot \tfrac{1}{4}\Big) =$$

$$= \left(1, \frac{2}{30};\ 2, \frac{2}{30} + \frac{8}{20};\ 4, \frac{2}{30};\ 3, \frac{8}{40};\ 5, \frac{8}{40}\right) =$$

$$= \left(1, \frac{1}{15};\ 2, \frac{4+24}{60};\ 4, \frac{1}{15};\ 3, \frac{1}{5};\ 5, \frac{1}{5}\right) =$$

$$= \left(1, \frac{1}{15};\ 2, \frac{7}{15};\ 4, \frac{1}{15};\ 3, \frac{1}{5};\ 5, \frac{1}{5}\right)$$

IT IS A LOTTERY SINCE $\frac{1}{15} + \frac{7}{15} + \frac{1}{15} + \frac{1}{5} + \frac{1}{5} = 1$ $\square$

2.5 TER

$$L^1 = \begin{array}{ll} -1 & 1/5 \\ 2 & 2/5 \\ 5 & 2/5 \end{array} \qquad L^2 = \begin{array}{ll} 2 & 1/2 \\ 3 & 1/4 \\ 4 & 1/4 \end{array}$$

HW

CHARACTERIZE $0{,}2\, L^1 + 0{,}8\, L^2$

THIS IS LOTTERY COMPOSITION WHICH IS THE ONLY OPERATION WHICH IS DONE ON PROBABILITIES AND NOT ON VALUES.

HOWEVER VALUES SHOULD BE EQUAL. IN ORDER TO HAVE THE SAME VALUES:

$$L^1 = (-1, 1/5;\ 2, 2/5;\ 5, 2/5;\ 3, 0;\ 4, 0)$$

$$L^2 = (-1, 0;\ 2, 1/2;\ 5, 0;\ 3, 1/4;\ 4, 1/4)$$

$$0{,}2\, L^1 + 0{,}8\, L^2 = \begin{array}{ll} -1 & 0{,}2/5 = 1/25 \\ 2 & 0{,}2 \cdot 2/5 + 0{,}8/2 = 2/25 + 2/5 = 12/25 \\ 5 & 0{,}2 \cdot 2/5 = 2/25 \\ 3 & 0{,}8 \cdot 1/4 = 1/5 \\ 4 & 0{,}8 \cdot 1/4 = 1/5 \end{array}$$

$\square$

2.6 REWRITE THE PREVIOUS LOTTERIES IN TERMS OF MONETARY OUTCOMES

- COIN FLIPPING. SUPPOSE I BET 1 EURO ON HEAD, WE CAN HAVE TWO LOTTERIES

HEAD: PAY 1 EURO, RECEIVE IT BACK PLUS ANOTHER EURO TAIL: PAY 1 EURO, RECEIVE NOTHING	HEAD: RECEIVE BACK MY 1 EURO PLUS ANOTHER ONE TAIL: DO NOT RECEIVE ANYTHING
$L = (+1, 50\%; -1, 50\%)$	$L = (+2, 50\%; 0, 50\%)$
IN THIS CASE 1 EURO IS INSERTED DIRECTLY IN THE LOTTERY OUTCOMES.	IN THIS CASE 1 EURO IS CONSIDERED A COST FOR PARTICIPATING TO A VERY CONVENIENT LOTTERY

- DIE ROLL. SUPPOSE I BET 1 EURO ON NUMBER 1 AND RECEIVE 10 TIMES MY BET BACK

$$L = (+9, 1/6; -1, 1/6; -1, 1/6; -1, 1/6; -1, 1/6; -1, 1/6) =$$

$$= (+9, 1/6; -1, 5/6)$$

MOST PEOPLE WANT TO PARTICIPATE IN THIS LOTTERY BECAUSE THEY FIND IT TO BE CONVENIENT. THIS IS BECAUSE THEY CALCULATE THE <u>EXPECTED VALUE</u> OF THE LOTTERY

$$E(L) = \sum_{i=1}^{5} x_i p_i = 9 \cdot \frac{1}{6} + (-1) \cdot \frac{5}{6} = \frac{4}{6} = \frac{2}{3}$$

AND SINCE IT IS POSITIVE THEY CONSIDER THIS LOTTERY TO BE CONVINIENT.

2.7

– ANOTHER DIE ROLL

WE BET 1 EURO, IF 6 COMES OUT WE GET 5 EURO, IF 5 COMES OUT WE GET 1 EURO, FOR OTHER NUMBERS NOTHI

$$= \left(-1,\ 4/6\ ;\ 0,\ 1/6\ ;\ +4,\ 1/6\right)$$

$$E(L) = -1 \cdot \frac{2}{3} + 0 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} = -\frac{2}{3} + \frac{4}{6} = -\frac{4}{6} + \frac{4}{6} = 0$$

MANY PEOPLE CONSIDER THIS LOTTERY TO BE INDIFFERENT

– ROUGE/NOIR ROULETTE

BET 1 EURO ON RED, RECEIVE 2 EURO IF RED, NOTHING OTHERWIS

$$L = \left(+1,\ 18/37\ ;\ -1,\ 19/37\right)$$

$$E(L) = 1 \cdot \frac{18}{37} + (-1) \cdot \frac{19}{37} = -\frac{1}{37}$$

HOWEVER MANY PEOPLE PLAY AT THE ROULETTE !!!

– ROUGE/NOIR WITH OTHER RULES

BET 1 EURO ON RED, RECEIVE 2 EURO IF RED, 0 IF BLACK, 1 IF Ø

$$L = \left(+1,\ 18/37\ ;\ -1,\ 18/37\ ;\ 0,\ 1/37\right)$$

$$E(L) = 0$$

2.8

- HOUSE FIRE DAMAGE

$$E(L) = -100'000 \cdot \frac{0,2}{100} + (-20000) \cdot \frac{0,8}{100} + 0 \cdot \frac{99}{100} =$$

$$= -200 - 160 + 0 = -360 \text{ €}$$

MANY PEOPLE PAY MONEY TO AVOID PARTICIPATING IN THIS LOTTERY, USUALLY MORE THAN THE EXPECTED VALUE

- NATIONAL LOTTERY TICKET

BUY 1 TICKET FOR 3 EUROS

$$L = \left(+1'000'000 - 3,\ 1/5'000'000;\ +100'000 - 3,\ 1/500'000;\ -3,\ \frac{4'999'989}{5'000'000}\right)$$

$$E(L) = \frac{999'997}{5'000'000} + \frac{99'997}{500'000} - 3 \cdot \frac{4'999'989}{5'000'000} =$$

$$= \frac{999'997 + 99'9970 - 3 \cdot 4'999'989}{5'000'000} \approx -2,6 \text{ €}$$

BUT MANY PEOPLE BUY A LOT OF LOTTERIES' TICKETS

PARADOXES

CONSIDERING ONLY $E(L)$, THERE ARE MANY PSYCHOLOGICAL PARADOXES:

1) MANY PEOPLE DO NOT PARTICIPATE IN $L = (+1,50\%; -1,50\%)$
2) MANY PEOPLE DO PARTICIPATE IN NATIONAL LOTTERIES
3) MANY PEOPLE BUY INSURANCES LARGER THAN $E(L)$ TO AVOID RISKING HOUSE FIRE

2.9

4) LOGICAL PARADOX: SANT PETERSBURG PARADOX

I FLIP A COIN UNTIL HEAD APPEARS. IF IT APPEARS THE FIRST TIME, I PAY 2 EURO, 4 IF IT APPEARS THE SECOND TIME, 8 THE THIRD, 16 THE FOURTH, 32 THE FIFTH, 64 THE SIXTH AND SO ON.
IF IT APPEARS AFTER 21 FLIPS, I WILL PAY 1'048'576 EURO

$$L = (2, 1/2;\ 4, 1/4;\ 8, 1/8;\ 16, 1/16;\ 32, 1/32;\ \dots)$$

$$E(L) = 2\cdot\frac{1}{2} + 4\cdot\frac{1}{4} + 8\cdot\frac{1}{8} + 16\cdot\frac{1}{16} + \dots = 1+1+1+1+\dots = +\infty$$

EXPECTED VALUE IS $+\infty$. IF YOU DO NOT LIKE INFINITE OUTCOMES, WE CAN STOP AT FLIP 1'000'000 AND LET THE GAMBLER WIN $2^{1'000'000}$ EURO. IN THIS CASE

$$L = (2, 1/2;\ 4, 1/4;\ \dots;\ 2^{999'999}, 1/2^{999'999};\ 2^{1'000'000}, \text{REST OF PROBABILITIES})$$

$$E(L) = 2\cdot\frac{1}{2} + 4\cdot\frac{1}{4} + \dots + 2^{999999}\cdot\frac{1}{2^{999999}} + 2^{1000'000}\cdot \text{REST OF PROBAB.} =$$

$$= 1+1+\dots + 1 + \text{SOMETHING LARGER THAN } 1 > 1'000'000\ €$$

IN THIS CASE $E(L)$ IS A VERY LARGE AMOUNT OF EURO!

HOWEVER, <u>NOBODY</u> WOULD PAY 500'000 € TO PARTICIPATE IN THIS LOTTERY, EVEN THROUGH $E(L) > 1000'000$ € !
PROBABLY NOT EVEN 500 € WOULD BE PAID!

THESE PARADOXES EXIST BECAUSE MONEY HAS A VALUE WHICH IS DIFFERENT FROM ITS FACE VALUE!