Risk attitude and lotteries.

U(E(L)) > E(U(L)) for every L → agent is strictly **risk-averse**
U(E(L)) < E(U(L)) for every L → agent is strictly **risk-lover**

U(E(L)) ≥ E(U(L)) for every L → agent is non strictly risk-averse
U(E(L)) ≤ E(U(L)) for every L → agent is non strictly risk- lover

U(E(L)) = E(U(L)) for every L → agent is **risk-indifferent**

The left side of these inequalities represent the utility of the expected value, therefore the utility of an average result. The right side represents instead applying the utility to the values of the lottery, and is therefore a measure of the utility of the lottery.
Therefore, agents preferring the expected value rather than the participation in the lottery are risk-averse and those preferring participating rather than receiving the expected value are risk-lover.

Note that inequalities must hold for every lottery. In the previous examples, U=ln (x+e)-1 or U=exp(x)-1, we proved the attitude only for the particular lottery L=(-1, 2/3; 0, 1/6; +4, 1/6) and not in general. Therefore, in these cases we say that the user dislikes/likes the lottery, but we may, for now, not say that the agent is risk-averse/lover.

It is important to note that a risk-averse agent does not refuse all the lotteries, while a risk-lover does not accept all the lotteries. Clearly, both will continue to refuse lotteries with only negative outcomes, such as house fire damage, and will continue to participate in lotteries with only positive outcomes, such as the Sant Petersburg lottery.
The difference lies in the fact that a risk-averse wants to pay less than the expected value to participate in a lottery, while the risk-lover wants to pay more than the expected value to participate in a lottery. Obviously, for lotteries with negative expected value, the situation is reversed: a risk-averse wants to pay more than the expected value to avoid a lottery, while the risk-lover wants to pay less than the expected value to avoid it. In summary, the risk-lover's opinion of the lottery's value is always higher than the expected value, while the risk-averse agent's opinion is always smaller.
Therefore, from now one we will talk about liking and disliking lotteries, with the meaning of:

- **like**: thinking that the lottery is worth more than its expected value
- **dislike**: thinking that the lottery is worth less than its expected value
- **indifferent**: not like nor dislike, lottery is worth its expected value

Therefore:

- a risk-averse agent dislikes all lotteries (even though he will still participate in convenient lotteries unless he is offered E(L) in return)
- a risk-lover agent likes all lotteries (even though he will still not participate in not convenient lotteries unless he is offered E(L) in return)
- a risk-indifferent agent is indifferent to all lotteries

Consider now the risk lover relation $\mathrm{U}(\mathrm{E}(L)) < \mathrm{E}(U(L))\ \forall L$. It can be written, when L is a discrete lottery, as

$$\mathrm{U}\left(\sum_{j=1}^{n} x_j p_j\right) < \left(\sum_{j=1}^{n} \mathrm{U}(x_j) p_j\right)\ \forall x_j\ \forall p_j \in [0;1] \text{ with } \sum_{j=1}^{n} p_j = 1.$$

Considering for simplicity the case $n = 2$ with only two outcomes where $p_2 = 1 - p_1$, we have

$$\mathrm{U}(x_1 p_1 + x_2(1-p_1)) < \mathrm{U}(x_1)p_1 + \mathrm{U}(x_2)(1-p_1)\ \forall x_1 \forall x_2\ \forall p_1 \in [0;1].$$

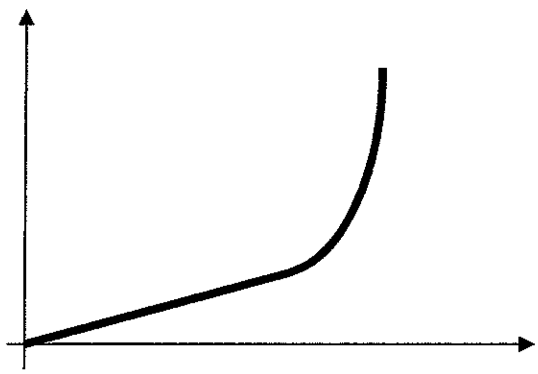
Now we recall the definition for a convex function which is

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)\ \forall x \forall y\ \forall \lambda \in [0;1],$$

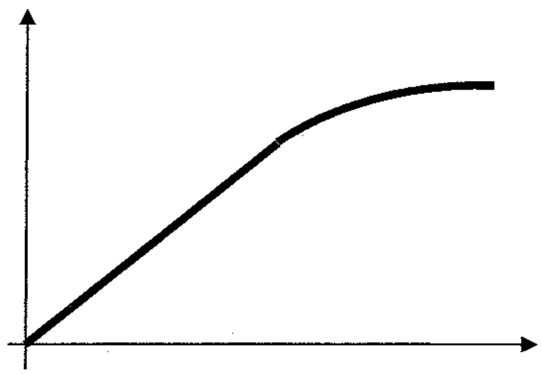
and therefore we can easily see that being a risk-lover agent is equivalent to having a convex utility function.

In the same way, a concave utility function identifies risk-averse agents, while a linear utility identifies risk indifferent agents.

If the utility is non strictly concave, meaning that it is linear in some intervals and concave in others, the agent will be non strictly risk-averse, meaning that he will dislike some lotteries and be indifferent to others, but never like any lottery. Vice versa for a non strictly convex function.



Non strictly convex function



Non strictly concave function

In order to check concavity or convexity of functions we can simply look at the graph, as it is suggested whenever the function definition is split, or calculate the second derivative.

4.6 CHECK THAT $\ln x$ IS A UTILITY FUNCTION
WHAT IS AGENT'S ATTITUDE TOWARDS RISK?
AND FOR $U = x^3$ AND $U = e^x$?

$U = \ln x$ $\quad U' = \frac{1}{x}$ $\quad$ FOR $x > 0$ IT IS $U' > 0 \Rightarrow U$ INCREASING

$U(0)$ IS NOT DEFINED

U IS A UTILITY FUNCTION FOR $x > 0$

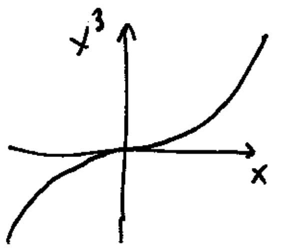
$U'' = -\frac{1}{x^2} < 0$ $\quad \Rightarrow U$ IS CONCAVE $\Rightarrow$ AGENT IS RISK-AVERSE

$U = x^3$ $\quad U' = 3x^2 > 0 \Rightarrow U$ IS INCREASING

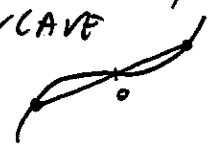
$U(0) = 0$ $\quad \Rightarrow U$ is A UTILITY FUNCTION $\forall x$

$U'' = 6x$ $\quad U'' > 0$ FOR $x > 0 \Rightarrow$ FOR $x > 0$ AGENT IS RISK-LOVER

$U'' < 0$ FOR $x < 0 \Rightarrow$ FOR $x < 0$ AGENT IS RISK-AVERSE



FOR $x = 0$ IT IS NEITHER STRAIGHT, NOR CONVEX, NOR CONCAVE

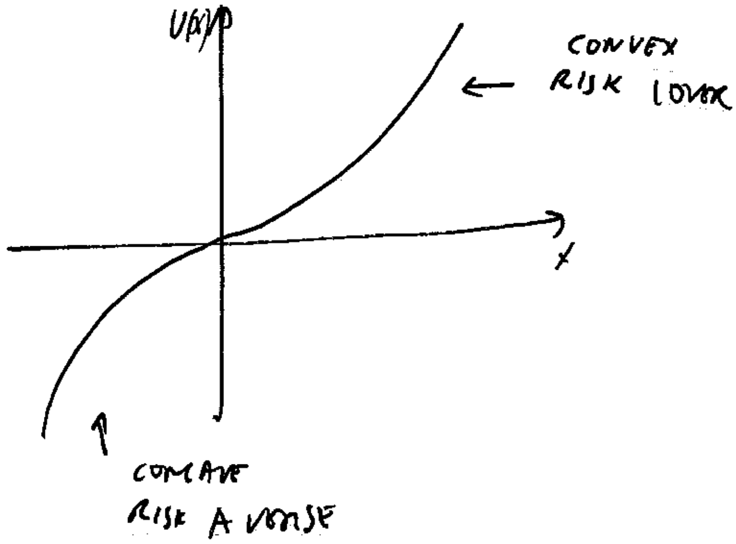


$U = e^x$ $\quad U' = e^x > 0 \ \forall x \Rightarrow U$ IS ALWAYS INCREASING

$U(0) = e^0 = 1$ $\quad U$ IS NOT "OFFICIALLY" A UTILITY FUNCTION.
HOWEVER $V(x) = e^x - 1$ IS AN OFFICIAL UTILITY FUNCTION

$U'' = e^x > 0 \Rightarrow$ $\quad \Rightarrow$ CONVEX $\Rightarrow$ RISK-LOVER AGENT $\forall x$

4.7 FOR $U(x) = x^3$ MAKE EXAMPLE OF LIKED, INDIFFERENT AND DISLIKED LOTTERIES



$L = (1, 50\% \,;\, 3, 50\%)$

$L = (-3, 50\% \,;\, -1\ 50\%)$

LET'S SEE $L = (-1, 50\% \,;\, +1, 50\%)$

$E(L) = 0 \qquad U(E(L)) = 0 \qquad E(U(L)) = (-1)^3 \cdot 50\% + (1)^3 \cdot 50\% = 0$

IT IS INDIFFERENT

CAN WE SAY THAT ALL LOTTERIES WHICH CROSS 0 ARE INDIFFERENT ?

NO, BECAUSE IF WE CHOOSE A NON-SYMMETRIC LOTTERY:

$L = (-1, 50\% \,;\, +2, 50\%) \qquad E(L) = -\frac{1}{2} + 1 = +\frac{1}{2} \quad U(E(L)) = \frac{1}{8}$

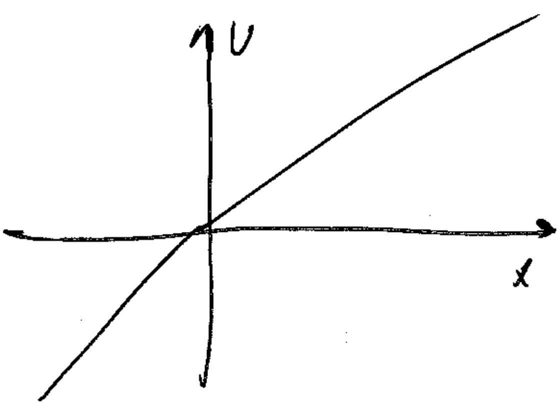
$E(U(L)) = (-1)^3 \cdot 50\% + 2^3 \cdot 50\% = -\frac{1}{2} + \frac{8}{2} = +\frac{7}{2}$

LOTTERY LIKED !

$\square$

4.8

$$U = \begin{cases} Ax & x \geq 0 \\ Bx & x < 0 \end{cases} \qquad A, B > 0$$



$$U' = \begin{cases} A > 0 & x < 0 \\ \text{DOES NOT EXIST} & x = 0 \\ B > 0 & x > 0 \end{cases}$$

IT IS INCREASING $\forall x \neq 0$

$U(0) = 0$

$\Downarrow$

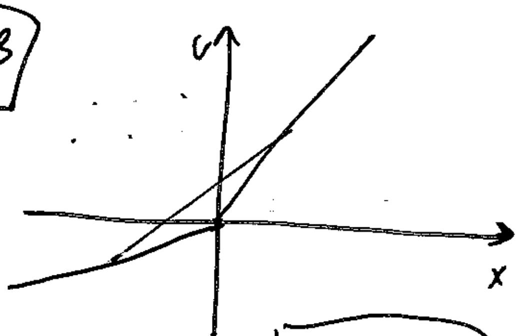
IT IS A UTILITY

LET'S SEE CONVEXITY AND CONCAVITY

$$U'' = \begin{cases} 0 & x < 0 \\ \text{DOES NOT EXIST} & x = 0 \\ 0 & x > 0 \end{cases}$$

IT IS LINEAR FOR $x < 0$ AND $x > 0$ $\Rightarrow$ AGENT IS RISK INDIFFERENT IN THOSE AREAS.

FOR $x = 0$ $\boxed{\text{IF } A > B}$

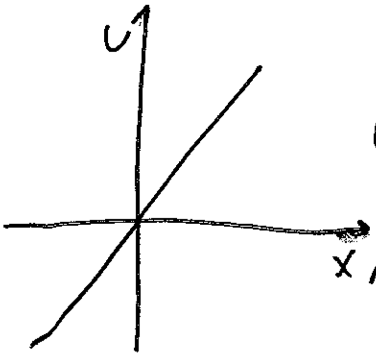


U IS CONVEX

$\Downarrow$

AGENT IS RISK LOVER IN $x = 0$

IF $\boxed{A = B}$



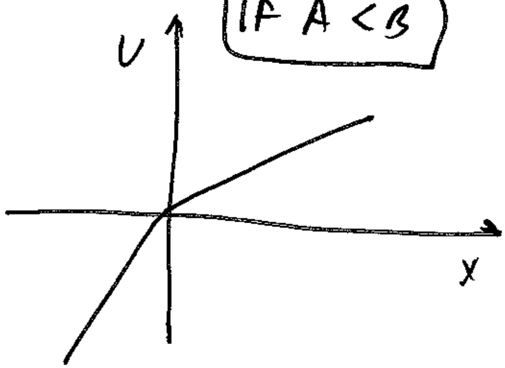
U IS LINEAR

$\Downarrow$

AGENT IS

X RISK INDIFFERENT

$\boxed{\text{IF } A < B}$



U IS CONCAVE

$\Downarrow$

AGENT IS RISK AVERSE IN $x = 0$

4.9

WHAT DOES IT MEAN

RISK INDIFFERENT FOR $x \neq 0$ AND

RISK AVERSE IN $x=0$? IT MEANS IN GENERAL NOT STRICTLY RISK AVERSE

CONSIDER, FOR EXAMPLE $A=2$ AND $B=1$. LOOK AT THESE LOTTERIES

* $L=(+2, 50\%; +4, 50\%)$

$E(L) = 2 \cdot 50\% + 4 \cdot 50\% = 1+2=3 \qquad U(E(L)) = 2 \cdot 3 = 6$

$E(U(L)) = U(2) \cdot 50\% + U(4) \cdot 50\% = 2 \cdot 2 \cdot 50\% + 2 \cdot 4 \cdot 50\% = 2+4=6$

$U(E(L)) = E(U(L)) \Rightarrow$ LOTTERY INDIFFERENT

* $L=(-1, 50\%; +1, 50\%)$

$E(L) = -\frac{1}{2} + \frac{1}{2} = 0 \qquad U(E(L)) = 0 \qquad E(U(L)) = U(-1) \cdot 50\% + U(1) \cdot 50\% =$

$= -1 \cdot 50\% + 2 \cdot 1 \cdot 50\% = -\frac{1}{2} + 1 = +\frac{1}{2}$

$E(U(L)) > U(E(L)) \Rightarrow$ LIKES THE LOTTERY

(THIS IS TRUE FOR ALL LOTTERIES WHICH CROSSES 0)

* $L=(-4, 50\%; -2\ 50\%)$

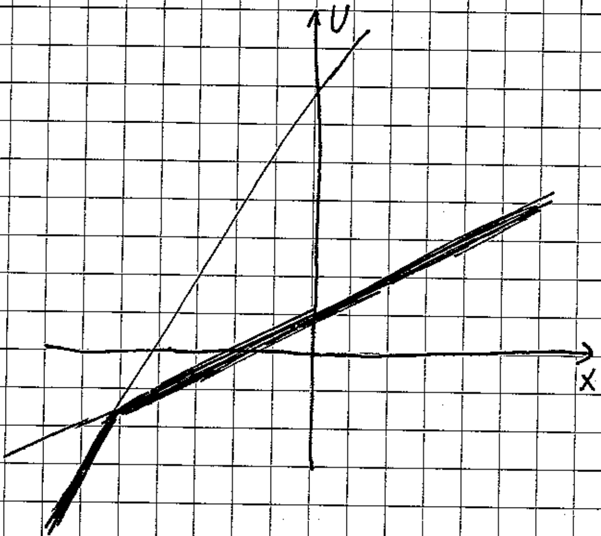
$E(L) = -4 \cdot 50\% - 2 \cdot 50\% = -2-1=-3 \qquad U(E(L)) = 1 \cdot (-3) = -3$

$E(U(L)) = U(-4) \cdot 50\% + U(-2) \cdot 50\% = -4 \cdot 50\% - 2 \cdot 50\% = -2-1=-3$

$U(E(L)) = E(U(L)) \Rightarrow$ L IS INDIFFERENT

4.10 $U(x) = \min(2x+1, 3x+7)$

IS AGENT RISK-AVERSE ?

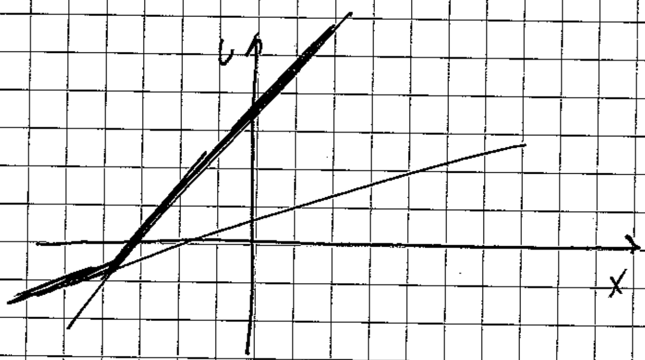


AGENT IS RISK-AVERSE BUT NOT STRICTLY.
HE DISLIKES SOME LOTTERIES AND IS INDIFFERENT TO OTHERS

$U(x) = \max(2x+1, 3x+7)$

IS AGENT RISK-AVERSE ?

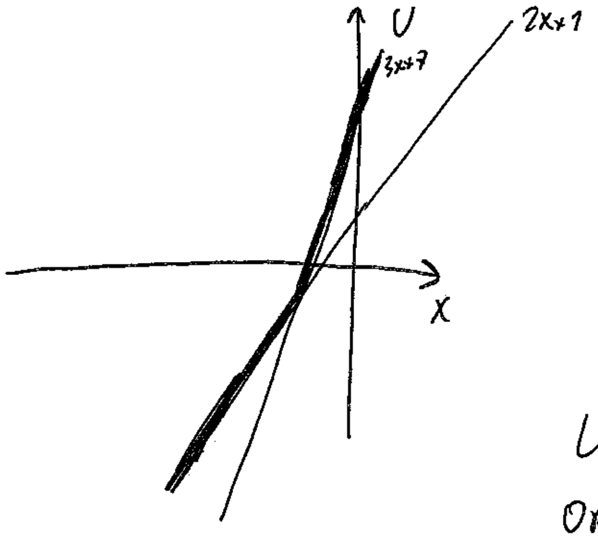
HW



AGENT IS RISK-LOVER, BUT NOT STRICTLY.
HE LIKES SOME LOTTERIES AND IS INDIFFERENT TO OTHERS.

4.10 BIS) $U = \max(2x+1, 3x+7)$

HW WHICH LOTTERIES (NON-DEGENERATE) ARE INDIFFERENT FOR THIS AGENT?



CROSSING POINT IS

$2x+1 = 3x+7$

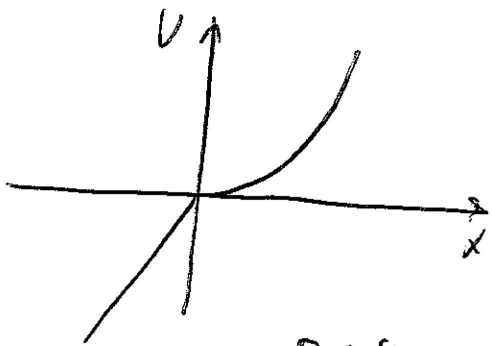
$-6 = x$

LOTTERIES WITH ALL VALUES SMALLER OR ALL VALUES LARGER THAN -6 ARE INDIFFERENT FOR THIS AGENT.

BECAUSE IN THESE AREAS UTILITY FUNCTION IS LINEAR AND THEREFORE $E[U(L)] = U(E(L))$

HW $U = \begin{cases} x^2 & x \geq 0 \\ 3x & x < 0 \end{cases}$ FIND AGENT ATTITUDE.

FIND SOME NON-DEGENERATED LOTTERIES WHICH ARE INDIFFERENT FOR THIS AGENT?



U FUNCTION IS NEITHER CONVEX NOR CONCAVE
$\Downarrow$
CANNOT SAY ANYTHING.

L FOR WHICH AGENT IS INDIFFERENT ARE FOR SURE THOSE WITH ONLY NEGATIVE VALUES.

NOTE: THERE MIGHT BE OTHER LOTTERIES WHICH CROSS Ø WHICH ARE INDIFFERENT TO THIS AGENT, BUT FOR SURE NOT L WITH POSITIVE OUTCOMES!

4.11

DEF: L IS <u>PURE-RISK</u> OR <u>ZERO-MEAN</u> WHEN $E(L)=0$

DEF: w_0 <u>WEALTH-LEVEL</u> IS THE AGENT'S WEALTH.

FROM NOW ON, WE WILL INCORPORATE IT INTO LOTTERY'S RESULT. THEREFORE

$$L \to L + w_0 \qquad (x_1, p_1; x_2, p_2; \dots) \to (x_1+w_0, p_1; x_2+w_0, p_2 \dots)$$

DEF: NEW EQUIVALENT DEFINITION FOR RISK-AVERSE

AN AGENT IS RISK-AVERSE

$\Updownarrow$

HE DISLIKES ALL PURE-RISK LOTTERIES AT ALL WEALTH-LEVELS

$\Updownarrow$

$$U(E(L+w_0)) > E(U(L+w_0)) \quad \forall L \text{ P.R. } \forall w_0$$

$$U(E(L+w_0)) = U(E(L)+w_0) = U(w_0)$$

THEREFORE $\boxed{U(w_0) > E(U(L+w_0)) \quad \forall L \text{ P.R. } \forall w_0}$

DEF: A_1 IS <u>NOT LESS RISK-AVERSE</u> THAN A_2

$\Updownarrow$

GIVEN A w_0 — A_1 DISLIKES ALL LOTTERIES DISLIKED BY A_2

$\Updownarrow$

GIVEN A w_0 — $U_2(w_0) > E(U_2(L+w_0)) \implies U_1(w_0) > E(U_1(L+w_0))$ FOR SAME L FOR SAME w_0

$\Updownarrow$

$$-\frac{U_1''}{U_1'} > -\frac{U_2''}{U_1'}$$ (WHEN U_1 AND U_2 ARE TWICE-DIFF)

DEF: <u>ARROW-PRAAT</u> DEGREE OF RISK-AVERSION $A(x) = -\frac{U''(x)}{U'(x)}$

NOTE: $A(x) < 0$ FOR RISK-LOVER, $A(x) > 0$ FOR RISK-AVERSE

4.12

EVALUATE A.P. FOR $U(x)=x^3$ $\underline{U(x)=\ln x}$ $\underline{U(x)=e^x-1}$ $\underline{U(x)=\begin{cases} Ax & x\geq 0 \\ Bx & x<0 \end{cases}}$ $A,B>0$ $U(x)=-e^{-\alpha x}$ $\alpha>0$

— = HOMEWORK

$U=x^3$ $U'=3x^2$ $U''=6x$ $-\frac{U''}{U'}=-\frac{6x}{3x^2}=-\frac{2}{x}$ $\boxed{x\neq 0}$

A.P. <0 $x>0$ A.P. >0 $x<0$

$U=\ln x$ $\boxed{x>0}$ $U'=\frac{1}{x}$ $U''=-\frac{1}{x^2}$ $-\frac{U''}{U'}=-\left(-\frac{1}{x^2}\right)/\frac{1}{x}=+\frac{x}{x^2}=+\frac{1}{x}$

A.P. >0

$U=e^x-1$ $U'=e^x$ $U''=e^x$ $-\frac{U''}{U'}=-1$ A.P. <0

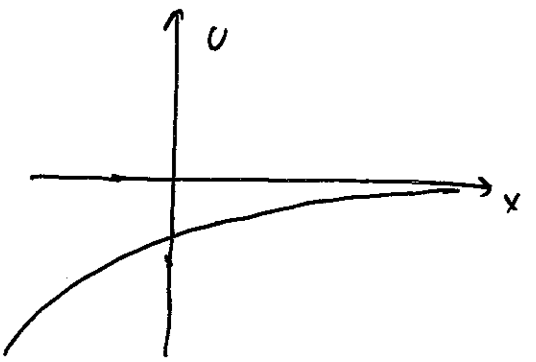
$U=\begin{cases} Ax & x\geq 0 \\ Bx & x<0 \end{cases}$ $U'=\begin{cases} A \\ \nexists \\ B \end{cases}$ $U''=\begin{cases} 0 & x\neq 0 \\ \nexists & x=0 \end{cases}$ $-\frac{U''}{U'}=0$ $x\neq 0$

$\nexists$ $x=0$

A.P. $=0$ FOR $x\neq 0$

$U=-e^{-\alpha x}$ $U'=+\alpha e^{-\alpha x}$ $U''=-\alpha^2 e^{-\alpha x}$ $-\frac{U''}{U'}=-\frac{-\alpha^2 e^{-\alpha x}}{\alpha e^{-\alpha x}}=+\alpha$

A.P. >0 $\forall x$



NOTE THAT IT IS NOT "OFFICIALLY" A UTILITY FUNCTION SINCE $U(0)\neq 0$. HOWEVER $U=-e^{-\alpha x}+1$ IS A PERFECT UTILITY FUNCTION AND ITS $U'=\alpha e^{-\alpha x}$ $U''=-\alpha^2 e^{-\alpha x}$ ARE THE SAME. □

4.13

$U_0(x)$ UTILITY FUNCTION OF RISK-LOVER, TWICE-DIF.

$U_1(x) = \exp(U_0(x))$

- IS $U_1(x)$ A UTILITY FUNCTION?
- WHAT CAN YOU SAY ABOUT AGENT 1?
- COMPARE A_0 AND A_1

$U_0(x)$ IS INCREASING
exp IS INCREASING $\Rightarrow \exp(U_0(x))$ IS INCREASING

$\Downarrow$

$U_1(x)$ IS UTILITY FUNCTION

NOTE THAT $U_1(0) = \exp(U_0(0)) = \exp(0) = 1$ BUT CONDITION $U_1(0)=0$ IS NOT IMPORTANT.

$U_0(x)$ IS CONVEX $\Rightarrow U_0'' > 0$

$$U_1'(x) = \exp(U_0(x)) \cdot U_0'(x)$$

$$U_1''(x) = \exp(U_0(x)) \cdot U_0'(x) \cdot U_0'(x) + \exp(U_0(x)) \cdot U_0''(x) =$$

$$= \underset{0}{\underset{\vee}{\exp(U_0(x))}} \Big[\underset{0}{\underset{\vee}{U_0'^{\,2}}} + \underset{0}{\underset{\vee}{U_0''}} \Big] > 0 \Rightarrow U_1'' > 0$$

$\Downarrow$

U_1 CONVEX

$\Downarrow$

AGENT 1 IS RISK-LOVER

$$A_0(x) = -\frac{U_0''(x)}{U_0'(x)} \qquad A_1(x) = -\frac{U_1''(x)}{U_1'(x)} = -\frac{\exp(U_0)\,(U_0'^{\,2} + U_0'')}{\exp(U_0)\,U_0'} = \frac{U_0'^{\,2} + U_0''}{U_0'} =$$

$$= -U_0'(x) - \frac{U_0''(x)}{U_0'(x)} = \underbrace{-U_0'(x)}_{\text{NEG OR ZERO}} + A_0(x) \Rightarrow A_1(x) \leq A_0(x)$$

$\square$