6.1 SMALL RISKS

L A PURE-RISK LOTTERY

$\rho^{\wedge} = \rho(k)$ IS RISK-PREMIUM FOR kL $\quad k \in \mathbb{R} \quad k \geq 0$

$$E(U(w_0 + kL)) = U(w_0 - \rho(k))$$

→ WE WANT NOW TO CALCULATE $\rho(0)$, $\rho'(0)$ AND $\rho''(0)$

$\rho(0)$ IS ZERO BECAUSE LOTTERY IS DETERMINISTIC $kL = (0, 100\%)$ WHEN $k=0$

LET'S CONSIDER THE DERIVATIVE OF $E(U(w_0 + kL))$

$$\frac{d}{dk} E(U(w_0 + kL)) = \frac{d}{dk} \sum_{i=1}^{n} U(w_0 + kx_i) \cdot p_i = \sum_{i=1}^{n} p_i \, U'(w_0 + kx_i) \cdot x_i =$$

$$= E(U'(w_0 + kL) \cdot L)$$

$$E(U'(w_0 + kL) \cdot L) = U'(w_0 - \rho(k))(-\rho'(k))$$

FOR $k=0$

$$E(U'(w_0) \cdot L) = U'(w_0)(-\rho'(0))$$

$$U'(w_0) \cdot E(L) = U'(w_0)(-\rho'(0)) \qquad U' > 0 \text{ SINCE INCREASING}$$

$$E(L) = -\rho'(0) \qquad E(L) = 0 \text{ SINCE IT IS PURE-RISK}$$

$$\Downarrow$$

$$\rho'(0) = 0$$

DERIVING AGAIN

$$E(U''(w_0 + kL) \cdot L^2) = U''(w_0 - \rho(k)) \cdot \rho'(k)^2 + U'(w_0 - \rho(k))(-\rho''(k))$$

FOR $k=0$

$$E(U''(w_0) \cdot L^2) = 0 + U'(w_0)(-\rho''(0))$$

$$\rho''(0) = -\frac{U''(w_0) \cdot E(L^2)}{U'(w_0)}$$

6.2 IN GENERAL, ANY FUNCTION $f(x)$ CAN BE APPROXIMATED

WITH: $f(x) \approx f(0) + f'(0) \cdot x + f''(0)\frac{x^2}{2}$ FOR X CLOSE TO 0

THIS IS CALLED <u>TAYLOR EXPANSION</u>

THEREFORE TAKING $\rho(K)$ FOR K CLOSE TO Ø

$$\rho(K) \approx \rho(0) + \rho'(0)\cdot K + \rho''(0)\cdot\frac{K^2}{2} = 0 + 0 - \frac{U''(w_0)\,E(L^2)}{U'(w_0)}\cdot\frac{K^2}{2}$$

$$\rho(K) \approx -\frac{U''(w_0)}{U'(w_0)}\cdot\frac{K^2\,E(L^2)}{2} = A(w_0)\cdot\frac{K^2 E(L^2)}{2}$$

NOTE THAT $E(L^2) - (E(L))^2 = VAR(L) \Rightarrow$ SINCE L PURE RISK
$E(L^2) - 0 = VAR(L)$

$$\boxed{\rho^{\Omega} = \rho(K) \approx A(w_0)\cdot\frac{K^2}{2}\,VAR(L) \qquad \text{FOR K CLOSE TO ZERO}}$$

→ PREMIUM AT RISK IS APPROXIMATELY PROPORTIONAL TO ARROW-PRATT DEGREE OF RISK AVERSION, TO THE VARIANCE OF THE ORIGINAL L (HOW SPREAD ARE ITS OUTCOMES), AND <u>QUADRATIC</u> PROPORTIONAL TO THE SIZE OF OUTCOMES

$$C^e(w_0, KL, U) = E(KL) - \rho^{\Omega}(w_0 + E(KL),\, KL - E(KL),\, U)$$

WITH L A GENERIC LOTTERY

$$C^e \approx K\,E(L) - A(w_0 + K\,E(L))\cdot\frac{K^2}{2}\cdot VAR(L - E(L))$$

BUT $VAR(L - COST) = VAR(L)$

$$\boxed{C^e \approx K\,E(L) - A(w_0 + K\,E(L))\cdot VAR(L)\cdot\frac{K^2}{2}}$$

<u>USUALLY</u> ALSO $A(w_0 + K\,E(L)) \approx A(w_0)$

6.3 EXERCISE: $L_1 = (1, 50\%; 0, 50\%)$

$L_2 = (1, 50\%; -1, 50\%)$

CALCULATE C^ℓ FOR kL_1 AND kL_2 WHEN k IS SMALL.

WHICH ONE IS SMALLER FOR A RISK-AVERSE AGENT?

$$C^\ell \approx E(L) \cdot k - A(w_0) \frac{k^2}{2} \, VAR(L)$$

WE USE THE FULL APPROXIMATION TO BE ABLE TO COMPARE THE TWO VALUES OF A

$E(L_1) = \frac{1}{2}$ $\quad E(L_2) = 0$

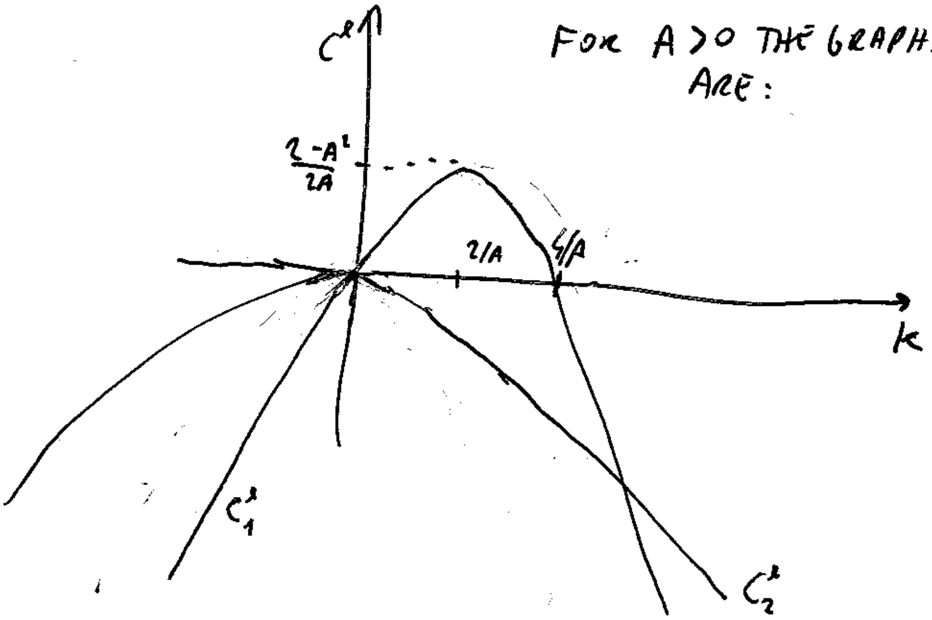
$$VAR(L_1) = E\left(\left(L_1 - \tfrac{1}{2}\right)^2\right) = 50\% \cdot \left(\tfrac{1}{2}\right)^2 + 50\% \cdot \left(-\tfrac{1}{2}\right)^2 = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$VAR(L_2) = E(L_2^2) = 50\% \cdot 1^2 + 50\% \cdot (-1)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$C_1^\ell \approx \frac{k}{2} - A(w_0) \frac{k^2}{2} \cdot \frac{1}{4} \qquad C_2^\ell \approx k \cdot 0 - A(w_0) \frac{k^2}{2}$$

$$\frac{k}{2} - A(w_0) \frac{k^2}{8} \qquad\qquad -\frac{k^2}{2} A(w_0)$$

FOR $A > 0$ THE GRAPHS ARE:



RESULT:

FOR SMALL VALUES OF $k > 0$: $C_2^\ell < C_1^\ell$

NOTE: WHAT WE HAVE DONE IS NOT PRECISE. C_1^ℓ SHOULD NOT BE APPROXIMATED WITH w_0 BUT WITH $w_0 + \frac{k}{2}$ AND THEREFORE

$C_1^\ell \approx \frac{k}{2} - A\left(w_0 + \frac{k}{2}\right) \frac{k^2}{8}$ BUT FOR k SMALL $A(w_0) \approx A\left(w_0 + \frac{k}{2}\right)$

□

6.4 $L_1 = (1, 0{,}5; -1, 0{,}5)$ $L_2 = (1, 0{,}4; 0, 0{,}2; -1, 0{,}4)$

- FOR A RISK-AVERSE, WHICH IS RISKIER FOR SMALL RISK?

HOMEWORK

WE ARE DEALING WITH PURE RISK SINCE

$E(L_1) = 0 \quad E(L_2) = 0$

THEREFORE WE CAN DECIDE WHICH IS RISKIER BY LOOKING AT WHICH L HAS A LARGER ρ^r (FOR RISK-AVERSE AGENTS, OBVIOUSLY!)

$$\rho^r = \frac{1}{2} A(w_0)\, k^2\, VAR(L)$$

$$\rho^r = +\frac{1}{2} A(w_0)\, k^2\, VAR(L)$$

$VAR(L_1) = 1^2 \cdot 0{,}5 + (-1)^2 \cdot 0{,}5 = 1$

$VAR(L_2) = 1^2 \cdot 0{,}4 + 0^2 \cdot 0{,}2 + (-1)^2 \cdot 0{,}4 = 0{,}8$

$\rho_1^r = \frac{A(w_0)}{2} k^2 \qquad \rho_2^r = \frac{A(w_0)}{2} k^2 \cdot 0{,}8$

SINCE THE AGENT IS RISK AVERSE $\Rightarrow A(w_0) > 0$

$\Downarrow$

L_1 IS RISKIER THAN L_2

$\square$

6.5 EVALUATE C^{ℓ} FOR kL, $L \sim N(0{,}2; 0{,}01)$,
WITH $U = -e^{-x}$ AND $W_0 = 1$

FOR SMALL RISK

$$C^{\ell} \approx E(L)k - \frac{1}{2} \cdot A(W_0 + kE(L)) \cdot k^2 VAR(L)$$

$E(L) = 0{,}2$ $\quad VAR(L) = VAR(N(0{,}2; 0{,}01)) = 0{,}01$

$$A(W_0) = -\frac{U''(W_0 + k0{,}2)}{U'(W_0 + k0{,}2)} = -\frac{-e^{-x}}{e^{-x}}\Bigg|_{\substack{\text{EVALUATED} \\ \text{IN } W_0 + k0{,}2}} = -\frac{-e^{-1-k0{,}2}}{e^{-1-k0{,}2}} = 1$$

$$C^{\ell} = 0{,}2\,k - \frac{k^2}{2}\,0{,}01$$

D

NOTE: SINCE $A(W_0) = 1$ $\forall W_0$ (IT IS CONSTANT),
C^{ℓ} CALCULATED WITH BOTH APPROXIMATIONS YEALDS
THE SAME RESULT $0.2k - \frac{k^2}{2} 0.01$

6.6

$L \sim N(0; \theta^2)$ $\quad U = -e^{-x}$ $\quad w_0$

FIND ρ^2 FOR kL FOR ANY k

FIND ρ^2 FOR kL FOR SMALL k

L IS PURE-RISK $\quad U(w_0 - \rho^2) = E(U(w_0 + kL))$

$$E(U(w_0+kL)) = \int_{-\infty}^{+\infty} -e^{-(w_0+ks)} \cdot \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{s^2}{2\theta^2}} ds = -e^{-w_0} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{ks2\theta^2+s^2}{2\theta^2}} ds$$

WE TRY TO BUILD A PERFECT SQUARE AT NUMERATOR

$(s+B)^2 = s^2 + 2Bs + B^2$ $\quad$ I TAKE $B = k\theta^2$

$(s+k\theta^2)^2 = s^2 + 2k\theta^2 s + k^2\theta^4$

$$= -e^{-w_0} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{(s+k\theta^2)^2 - k^2\theta^4}{2\theta^2}} ds = -e^{-w_0} \cdot e^{+\frac{k^2\theta^4}{2\theta^2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{(s+k\theta^2)^2}{2\theta^2}} ds =$$

$$= -e^{-w_0 + \frac{k^2\theta^2}{2}} \int_{-\infty}^{+\infty} f_{N(-k\theta^2;\theta^2)}(s)\, ds = -e^{-w_0 + \frac{k^2\theta^2}{2}}$$

$U(w_0 - \rho^2) = -e^{-w_0+\rho^2}$ $\quad -e^{-w_0+\rho^2} = -e^{-w_0+\frac{k^2\theta^2}{2}}$

$e^{-w_0+\rho^2} = e^{-w_0+\frac{k^2\theta^2}{2}}$ $\quad -w_0+\rho^2 = -w_0 + \frac{k^2\theta^2}{2}$ $\quad \boxed{\rho^2 = \frac{k^2\theta^2}{2}}$

$$\rho^2 \approx A(w_0)\, \frac{k^2}{2} \cdot VAR(L) = \left(-\frac{-e^{-w_0}}{+e^{-w_0}}\right) \cdot \frac{k^2}{2} \cdot \theta^2 = \frac{k^2\theta^2}{2}$$

THE APPROXIMATION IS EQUAL TO THE NOT-APPROXIMATED VALUE.

THIS IS TRUE WHENEVER NOT-APPROXIMATED VALUE IS A SECOND DEGREE POLYNOMIAL.

6.7

$U = -e^{-x}$ $\quad w_0 = 2$ $\quad L = \text{UNIFORM}(-1, 1)$

HOMEWORK FIND $\rho^\wedge$ FOR KL: FOR EVERY K AND FOR SMALL K

FOR SMALL K: $\rho^\wedge \approx A(w_0) \cdot \frac{K^2}{2} \, VAR(L)$

$$A(w_0) = -\frac{U''(w_0)}{U'(w_0)} = -\frac{-e^{-w_0}}{+e^{-w_0}} = 1$$

$$VAR(L) = E\left((L - E(L))^2\right) = E(L^2) = \int_{-1}^{1} \frac{1}{2} \cdot s^2 \, ds = \left[\frac{s^3}{6}\right]_{-1}^{1} = \frac{1}{6} - \frac{-1}{6} = \frac{2}{6} = \frac{1}{3}$$

$$\rho^\wedge \approx K^2 \cdot \frac{1}{6}$$

FOR ANY K: $U(w_0 - \rho^\wedge) = E\left(U(w_0 + KL)\right)$

$$-e^{-2+\rho^\wedge} = E\left(U(w_0 + KL)\right) = \int_{-1}^{1} \frac{1}{2} \cdot \left(-e^{-Ks - w_0}\right) ds = \int_{-1} -\frac{1}{2} \, e^{-2-Ks} \, ds =$$

$$= -\frac{1}{2} \cdot e^{-2} \left[-\frac{e^{-Ks}}{K}\right]_{-1}^{1} = +\frac{e^{-2}}{2K}\left[e^{-Ks}\right]_{-1}^{1} = \frac{e^{-2}}{2K}\left(e^{-K} - e^{K}\right)$$

APPLYING ln ON BOTH SIDES

$$-2 + \rho^\wedge = \ln \frac{e^{-2}}{2K}\left(e^{K} - e^{-K}\right) \qquad \rho^\wedge = 2 + (-2) + \ln\left(e^{K} - e^{-K}\right) - \ln 2K$$

$$\rho^\wedge = \ln \frac{e^{K} - e^{-K}}{2K}$$

$\square$

6.8 $L = (-100, 50\% ; +100, 50\%)$

ρ^r FOR A IS 20 ρ^r FOR B IS 30

W_0 IS 0 FOR A AND B.

WHO IS MORE RISK-AVERSE

L IS A PURE-RISK LOTTERY, A LOTTERY WHICH ON AVERAGE PRODUCES A 0 RESULT AND WHICH CONTAINS ONLY RISK.

A AND B HAVE A POSITIVE ρ^r, MEANING THAT THEY ARE RISK-AVERSE, FOR THIS L AND W_0.

ρ^r_B IS LARGER, BECAUSE B IS MORE AFRAID OF RISK.

B IS, FOR THIS L AND THIS W_0, MORE RISK-AVERSE

IN GENERAL, WE CANNOT SAY THAT B IS MORE RISK-AVERSE

HOMEWORK

$L = (1000, 50\% ; 0, 50\%)$

FOR AGENT A	$C^e_A = 450$	$W_{0A} = 0$
FOR AGENT B	$C^e_B = 300$	$W_{0B} = 0$

WHO IS MORE RISK AVERSE?

THIS IS A LOTTERY WHERE YOU CAN NEVER LOSE. THEREFORE C^e IS ALWAYS ≥ 0 FOR EVERY AGENT.

AGENT B IS READY TO PAY LESS TO HAVE THE CHANCE TO WIN, THEREFORE HE IS MORE RISK-AVERSE. FOR THIS L.

IN GENERAL, WE CANNOT SAY THAT B IS MORE RISK-AVERSE