7.0 INSURANCES

BUYING AN INSURANCE MEANS CHANGING THE ORIGINAL LOTTERY L, WITH OUTCOMES ≤ 0, TO A <u>DETERMINISTIC</u> LOTTERY WHERE THE ONLY OUTCOME IS THE PAID <u>PREMIUM</u>

$$L^{\text{INSURED}} = (-\text{PREMIUM}, 100\%)$$

FOR EXAMPLE, $L = (-100000, 0{,}2\%; -20000, 0{,}8\%; 0, 99\%)$ BECOMES AFTER INSURANCE CONTRACT $L^{\text{INS.}} = (-\text{PREMIUM}, 100\%)$ AND THE INSURANCE COMPANY WILL TAKE ON ITSELF THE RISK

$$L^{\text{FOR INSURANCE COMPANY}} = (-100000 + \text{PREMIUM}, 0{,}2\%; -20000 + \text{PREMIUM}, 0{,}8\%; +\text{PREMIUM}, 99\%)$$

HOWEVER IN THIS WAY THE CUSTOMER WILL NOT DO ALL ITS BEST TO AVOID THE DAMAGE. TO FORCE HIM TO COLLABORATE, THE INSURANCE COMPANY ASKS HIM TO <u>SHARE THE RISK</u> THROUGH THESE CONTRACTS:

<u>PROPORTIONAL INSURANCE</u>: A FRACTION K OF LOSS IS PAID BY CUSTOMER

$$L^{\text{PROP. INS.}} = (-\text{PREMIUM} - 100000k, 0{,}2\%; -\text{PREMIUM} - 20000k, 0{,}8\%; -\text{PREMIUM}, 99\%)$$

$k \in (0;1)$

<u>STOP-LOSS INSURANCE</u>: CUSTOMER PAYS ALL LOSSES UP TO r (CALLED DEDUCTIBLE)

SUPPOSING $r = 30000$

$$L^{\text{STOP LOSS}} = (-\text{PREMIUM} - 30000, 0{,}2\%; -\text{PREMIUM} - 20000, 0{,}2\%; -\text{PREMIUM}, 99\%)$$

<u>LIMITED COVERAGE INSURANCE</u>: CUSTOMER PAYS ALL LOSSES EXCEEDING A LIMIT. SUPPOSING LIMIT $= 40000$

$$L^{\text{LIMIT. COV.}} = (-\text{PREMIUM} - 60000, 0{,}2\%; -\text{PREMIUM}, 0{,}2\%; -\text{PREMIUM}, 99\%)$$

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INSURANCES

L IS A LOTTERY (POSITIVE) WHICH DEALS WITH A LOSS

SOME PREMIUM FEATURES Π_L

1) NON NEGATIVE LOADING $\Pi_L \geq E(L)$
2) ADDITIVITY $\Pi_{L+L'} = \Pi_L + \Pi_{L'}$ FOR L AND L' INDIPENDENT
3) SCALE INVARIANCE $\Pi_{aL} = a\Pi_L \quad \forall a > 0$
4) CONSISTENCY $\Pi_{L+c} = \Pi_L + c \quad \forall c > 0$
5) NO RIPOFF $L \leq c_{max} \Rightarrow \Pi_L \leq c_{max}$

EXAMPLE A : PURE PREMIUM $\Pi_L = E(L)$

THE FIVE FEATURES ARE SATISFIED

1) OK
2) $\Pi_{L+L'} = E(L+L') = E(L) + E(L') = \Pi_L + \Pi_{L'}$ OK
3) $\Pi_{aL} = E(aL) = aE(L) = a\Pi_L$ OK
4) $\Pi_{L+c} = E(L+c) = E(L) + c = \Pi_L + c$ OK
5) $L \leq c_{max} \Rightarrow$ OUTCOMES $\leq c_{max}$

$\Pi_L = E(L) = \sum_{j=1}^{n} x_j p_j \leq$

$\leq \sum_{j=1}^{n} c_{max} p_j = c_{max} \sum_{j=1}^{n} p_j =$

$= c_{max}$

THIS MEANS $\Pi_L \leq c_{max}$ OK

EXAMPLE B : LOADED PREMIUM

$$\Pi_L = (1+b)E(L) + d \qquad b, d \geq 0$$

b ARE THE VARIABLE COSTS, d THE FIXED COSTS

1) $\Pi_L = (1+b)E(L) + d = E(L) + (bE(L) + d) > E(L)$ OK
2) $\Pi_{L+L'} = (1+b)\,E(L+L') + d = (1+b)\,E(L) + (1+b)E(L') + d =$

$\Pi_L + \Pi_{L'} = (1+b)\,E(L) + d + (1+b)\,E(L') + d = (1+b)\,E(L) + (1+b)\,E(L') + 2d$

OK ONLY WHEN $d = 0$

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3) HW $\Pi_{aL} = (1+b)E(aL)+d = a(1+b)E(L)+d$

$$a\Pi_L = a\left[(1+b)E(L)+d\right] = a(1+b)E(L)+a\cdot d$$

OK IF $d=0$

4) HW $\Pi_{L+c} = (1+b)E(L+c)+d = (1+b)E(L)+c+bc+d =$

$\Pi_L + c = (1+b)E(L)+d+c$ OK ONLY WITH $b=0$

5) $L \le c_{max} \Rightarrow E(L) \le c_{max}$ $\quad (1+b)E(L)+d \le c_{max}(1+b)+d$

OK IF $b=0$ AND $d=0$

IN OTHER CASES WE ARE NOT ABLE TO PROVE. A COUNTEREXAMPLE is $b=1$ $d=20$ $L=(1, 50\%; 0, 50\%)$

SO THE LOADED PREMIUM

- WITH $c=0$, $b>0$ DOES NOT SATISFY CONSISTENCY AND NORIPOFF
- WITH $b=0$, $c>0$ DOES NOT SATISFY ADDITIVITY, SCALE INV. AND NO RIPOFF

BUT ITS MAJOR PROBLEM IS THAT FOR $L=(2\ 50\%; 0\ 50\%)$ AND $L=(100\ 1\%; 0\ 99\%)$ IT HAS THE SAME PREMIUM.

EXAMPLE C: <u>VARIANCE PRINCIPLE</u>

$$\Pi_L = E(L) + b\,VAR(L) \qquad b>0$$

1) $\Pi_L = E(L)+b\,VAR(L) \ge E(L)$ OK

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2) $\Pi_{L+L'} = E(L+L') + b\,VAR(L+L') = E(L) + E(L') + b\,VAR(L) + b\,VAR(L')$

SINCE L AND L' INDEPENDENT

$\Pi_L + \Pi_{L'} = E(L) + b\,VAR(L) + E(L') + b\,VAR(L')$ OK

- OK SINCE L AND L' ARE INDEPENDENT

3) $\Pi_{aL} = E(aL) + b\,VAR(aL) = aE(L) + b\,a^2\,VAR(L) =$

$a\Pi_L \neq a[E(L) + b\,VAR(L)] = aE(L) + ab\,VAR(L)$ NO SINCE $b > 0$

4) HW $\Pi_{L+c} = E(L+c) + b\,VAR(L+c) = E(L) + c + a\,VAR(L) = \Pi_L + c$ OK

5) $L \leq c_{MAX} \Rightarrow E(L) \leq c_{MAX}$ BUT WE MAY NOT SAY THE SAME ON $VAR(L)$!

WE ARE NOT ABLE TO PROVE IT. A GOOD COUNTER-EXAMPLE IS FOR $b = 20$ $L = (1, 50\%; 0, 50\%)$

DOES NOT SATISFY SCALE INVARIANCE AND RIPS OFF

EXAMPLE D: STANDARD DEVIATION PRINCIPLE

$$\Pi_L = E(L) + b\sqrt{VAR(L)}$$

1) HW OK 2) HW $\Pi_{L+L'} = E(L) + E(L') + b\sqrt{VAR(L) + VAR(L')}$ ...

IN ANY CASE 2 IS NOT SATISFIED

3) HW $\Pi_{aL} = aE(L) + b\sqrt{VAR(aL)} = aE(L) + b\sqrt{a^2 VAR(L)} = a(E(L) + b\,VAR(L)) =$

$= a\Pi_L$

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4) $\Pi_{L+C} = E(L+C) + \alpha\sqrt{VAR(L+C)} = E(L)+C+\alpha\sqrt{VAR(L)} =$
HW

$= \Pi_L + C$

5) $L \leq C_{MAX} \Rightarrow E(L) \leq C_{MAX}$ BUT WE CANNOT SAY ANYTHING ON $VAR(L)$ NOR $\sqrt{VAR(L)}$
HW

WE ARE NOT ABLE TO PROVE IT. A COUNTEREXAMPLE IS
$b=100$ $L=(1, 50\%; 0, 50\%)$.

EXAMPLE E: PRINCIPLE OF ZERO UTILITY

$$U(w) = E(U(w+\Pi_L - L))$$

WHERE w IS THE INSURER SURPLUS AND U ITS UTILITY FUNCTION

SUBCASE: EXPONENTIAL PRINCIPLE

IF $U=-e^{-bx}$ $b>0$ IT IS THE EXPONENTIAL PRINCIPLE

$$\cancel{-e^{-bw}} = E\left(-e^{-b(w+\Pi_L-L)}\right) = E\left(-e^{-bw}\cdot e^{-b\Pi_L}\cdot e^{-bL}\right) = \cancel{-e^{-bw}}\cdot e^{-b\Pi_L} E\left(e^{-bL}\right)$$

$$\Downarrow$$

$$1 = e^{-\Pi_L b}\cdot E(e^{bL}) \qquad e^{\Pi_L b} = E(e^{bL}) \qquad \Pi_L = \frac{1}{b}\ln E(e^{bL})$$

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1) IF THE INSURER IS RISK-AVERSE $E(U(w+\Pi_L-L)) \leq U(w+\Pi_L-E(L))$

$U(w) = E(U(w+\Pi_L-L)) \leq U(w+\Pi_L-E(L))$

SINCE U IS INCREASING

$w \leq w+\Pi_L - E(L) \Rightarrow E(L) \leq \Pi_L$ OK

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2) FOR EXPONENTIAL

$$\Pi_{L+L'} = \frac{1}{b} \ln E\left(e^{bL+bL'}\right) = \frac{1}{b} \ln E\left(e^{bL} \cdot e^{bL'}\right) =$$

SINCE L AND L' INDEPENDENT $\Rightarrow$ e^{bL} AND $e^{bL'}$ INDEPENDENT

$$= \frac{1}{b} \ln E\left(e^{bL}\right) \cdot E\left(e^{bL'}\right) = \frac{1}{b} \ln E\left(e^{bL}\right) + \frac{1}{b} \ln E\left(e^{bL'}\right) =$$

$$= \Pi_L + \Pi_{L'}$$ <u>OK</u>

IN GENERAL IT DOES <u>NOT</u> WORK FOR ZERO UTILITY!

3) SCALE INVARIANCE. IT DOES NOT WORK IN GENERAL NOR FOR THE PARTICULAR CASE OF EXPONENTIAL PRINCIPLE

COUNTER-EXAMPLE $\quad U = -e^{-bx} \quad L = N(\mu, \theta^2)$

TAKE $b = 1$

$$\Pi_{\alpha L} = \ln E\left(e^{\alpha \cdot N(\mu,\theta^2)}\right) = \ln E\left(e^{N(\alpha\mu, \alpha^2\theta^2)}\right) =$$

BECAUSE $\alpha \cdot N$ IS STILL A NORMAL WITH $\alpha\mu$ EXP. VALUE AND $\alpha^2\theta^2$ VARIANCE

IT IS A LOGNORMAL

$$= \ln\left(e^{\alpha\mu + \frac{\alpha^2\theta^2}{2}}\right) = \alpha\mu + \frac{\alpha^2\theta^2}{2}$$

$$\alpha \Pi_L = \alpha \ln E\left(e^{N(\mu,\theta^2)}\right) = \alpha \cdot \left(\mu + \frac{\theta^2}{2}\right)$$ <u>NO</u>

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5) $L \le C_{max}$ $\quad -L \geqslant -C_{max}$

$$U(W) = E(U(W+\Pi_L - L)) \underset{\substack{\downarrow \\ U \text{ IS} \\ \text{INCREASING}}}{\geqslant} E(U(W+\Pi_L - C_{max})) = U(W+\Pi_L - C_{max})$$

$\Downarrow$ SINCE U IS INCREASING

$W \geqslant W+\Pi_L - C_{max} \Rightarrow \Pi_L \le C_{max}$ ok

4) CONSISTANCY

FOR Π_L $\quad U(W) = E(U(W+\Pi_L - L))$

FOR Π_{L+C} $\quad U(W) = E(U(W+\Pi_{L+C} - (L+C)))$

$$E(U(W+\Pi_L - L)) = E(U(W+\Pi_{L+C} - (L+C))) =$$

$$= E(U(W+\Pi_{L+C} - C - L)) \quad \forall W$$

$\Downarrow$

$W+\Pi_L - L = W+\Pi_{L+C} - C - L$

$\Downarrow$

$\Pi_L + C = \Pi_{L+C}$