8.1 EXP. UTILITY THEORY

DARA : DECREASING ABSOLUTE RISK AVERSION

IT IS A UTILITY FUNCTION THAT EXHIBITS DECREASING RISK AVERSION WHEN w_0 INCREASES

FOR DARA:

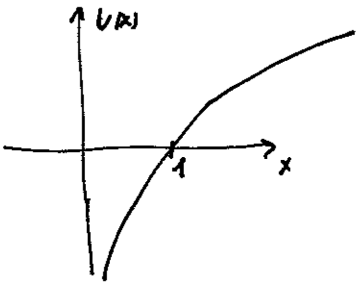
ρ^π is DIFFERENT. $\Rightarrow \dfrac{d\rho^\pi}{dw_0} < 0 \quad \forall L$

THEOREM: IF ρ^π IS 3-TIMES DIFFER. $\dfrac{dA(w_0)}{dw_0} < 0 \Rightarrow \dfrac{d\rho^\pi}{dw_0} < 0 \quad \forall L$

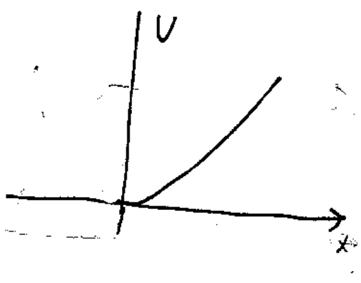
THIS MEANS THAT THE RICHER WE ARE, THE LESS RISK AVERSE WE ARE

CRRA : CONSTANT RELATIVE RISK AVERSION

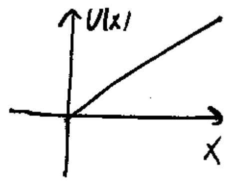
$U(x) = \ln x \qquad U(x) = \dfrac{x^{1-\gamma}}{1-\gamma} \quad \gamma \neq 1$ $\boxed{\text{FOR } X>0}$



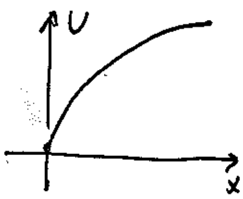
$\gamma < 0$



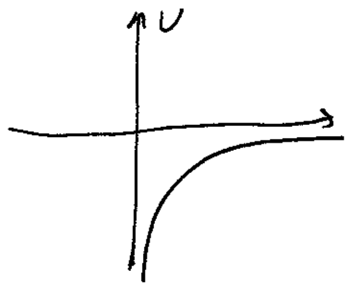
$\gamma = 0$



$0 < \gamma < 1$



$\gamma > 1$



8.2

THESE FUNCTIONS ARE ALSO DARA WHEN $\gamma > 0$:

$$* \quad A(x) = -\frac{U''(x)}{U'(x)} \qquad U = \ln x \quad A(x) = -\frac{-\frac{1}{x^2}}{\frac{1}{x}} = \frac{1}{x^2} x = \frac{1}{x}$$

$$A'(w_0) = -\frac{1}{w_0^2} < 0 \Rightarrow \frac{d\rho^*}{dw_0} < 0 \qquad \text{OK FOR } \ln x$$

$$* \quad A(x) = -\frac{U''(x)}{U'(x)} \qquad U = \frac{x^{1-\gamma}}{1-\gamma} \quad A(x) = -\frac{-\gamma x^{-\gamma-1}}{(1-\gamma)\frac{x^{-\gamma}}{1-\gamma}} = \gamma \frac{x^{-\gamma-1}}{x^{-\gamma}} = \frac{\gamma}{x}$$

$$A'(w_0) = -\frac{\gamma}{w_0^2} < 0 \text{ FOR } \gamma > 0 \Rightarrow \frac{d\rho^*}{dw_0} < 0 \qquad \text{OK FOR } \frac{x^{1-\gamma}}{1-\gamma}$$

WE CALL $R(x) = x\,A(x)$ THE COEFFICIENT OF RELATIVE RISK AVERSION

FOR THESE FUNCTIONS IT IS γ (WITH $\gamma = 1$ FOR $\ln x$)

CARA : CONSTANT ABSOLUTE RISK AVERSION

$$U(x) = \frac{e^{Ax}}{A} \qquad \forall A \in \mathbb{R}$$

$$U'(x) = e^{Ax} \qquad U''(x) = A e^{Ax} \qquad A(x) = -A$$

$$R(x) = -Ax$$

8.2 BIS

FIND COEFF. RELATIVE RISK AVERSION

FOR $U(x) = \int_0^x \exp\left(-\frac{t^2}{2}\right) dt \quad x \geq 0$

$U'(x) = \exp\left(-\frac{x^2}{2}\right) \qquad U''(x) = \exp\left(-\frac{x^2}{2}\right) \cdot -\frac{2x}{2}$

$A(x) = -\frac{U''(x)}{U'(x)} = -\frac{e^{-\frac{x^2}{2}} \cdot (-x)}{e^{-\frac{x^2}{2}}} = x$

$R(x) = A(x) \cdot x = x^2$

D

8.3 PROBLEMS WITH EXPECTED UTILITY THEORY

① ALLAIS - SAVAGE:

$$L_1 = \begin{cases} 30'000 & 33\% \\ 25'000 & 66\% \\ 0 & 1\% \end{cases} \qquad L_2 = 25'000 \quad 100\%$$

MANY PEOPLE SELECT L_2

$$L_3 = \begin{cases} 30'000 & 33\% \\ 0 & 67\% \end{cases} \qquad L_4 = \begin{cases} 25'000 & 34\% \\ 0 & 66\% \end{cases}$$

ALMOST EVERYBODY SELECTS L_3 !

THIS IS IN CONTRADICTION WITH UTILITY THEORY BECAUSE

PREFER L_2 OVER L_1 $\Rightarrow E(U(L_1)) < E(U(L_2)) \Rightarrow 0,33 \cdot U(30'000) +$

$+ 0,66 \cdot U(25'000) < U(25'000) \Longrightarrow 0,33\ U(30'000) < 0,34\ U(25'000)$

PREFER L_3 OVER L_4 $\Rightarrow E(U(L_3)) > E(U(L_4)) \Rightarrow 0,33 \cdot U(30'000) > 0,34\ U(25'000)$

CONTRADICTION !

② PEOPLE BUY INSURANCES $\Rightarrow$ THEY ARE RISK-AVERSE

BUT PEOPLE PREFER LIMITED COVERAGE TO A STOP LOSS $\Rightarrow$

THEY PREFER BIG UNPROBABLE RISKS TO SMALL PROBABLE RISKS $\Rightarrow$

THEY ARE RISK-LOVERS, AT LEAST FOR LARGE AMOUNTS.

8.4

③ PROBABILISTIC INSURANCE

$$L = \left(0 \,,\ 1-p \,;\ -\text{LOSS} \,,\ p \right)$$

YOU PAY A FRACTION r OF THE PREMIUM AND, IF THE BAD EVENT HAPPENS, YOU HAVE PROBABILITY r TO PAY THE REMAINING FRACTION AND BEING REFUNDED, AND 1 − FRACTION − PROBABILITY OF NOT BEING REFUNDED AND GETTING BACK YOUR PAID PREMIUM

$$L^{\text{PROB.}} = \left(-\text{PREMIUM} \cdot r \,,\ 1-p \,;\ -\text{PREMIUM} \,,\ pr \,;\ -\text{LOSS} \,,\ (1-r) \cdot p \right)$$

WE CAN SHOW THAT IF SOMEBODY IS INDIFFERENT BETWEEN L AND $L^{\text{FULL INSUR.}} = \left(-\text{PREMIUM} \,,\ 1 \right)$, THEN IT SHOULD PREFER, IF U IS CONCAVE, $L^{\text{PROB.}}$ TO $L^{\text{FULL INS.}}$

HOWEVER MANY PEOPLE DISLIKE PROBABILISTIC INSURANCES

8.5 BEYOND EXPECTED UTILITY

CERTAINTY: PREFER CERTAIN OUTCOMES, ESPECIALLY WHEN POSITI

REFLECTION: CHANGING THE RESULTS' SIGN, CHANGES THE LOTTERY'S PREFERENCE

$\Downarrow$

RISK-AVERSION IN POSITIVE DOMAIN IS RISK-SEEKING IN NEGATI

FOR EXAMPLE: $L^1 = (+10, 100\%)$ $L^2 = (0, 90\%; +100, 10\%)$

$L^{-1} = (-10, 100\%)$ $L^{-2} = (0, 90\%; -100, 10\%)$

IF AN AGENT PREFERS L^1 OVER L^2, THEN HE PREFERS L^{-2} OVER L^{-1}

CODING: PEOPLE DO NOT USE THEIR WEALTH AS W_0, BUT THEIR ASPIRATION OR THEIR EXPECTED WEALTH AS W_0. OR EVEN THEY USE ONLY THE EXPECTED INCOME AS W_0

FOR EXAMPLE, HAVING JUST LOST 2000 AND CHOOSING BETWEEN $L^1 = (+1000, 1)$ $L^2 = (+2000, 50\%; 0, 50\%)$

OFTEN THE SELECTION IS L^2 SINCE, IF THE LOSS IS NOT YET PSYCHOLOGICALLY ACCEPTED, IT IS MENTALLY AUTOMATICALLY CONVERTED TO

$\tilde{L}^1 = (-1000, 1)$ $\tilde{L}^2 = (0, 50\%; -2000\ 50\%)$

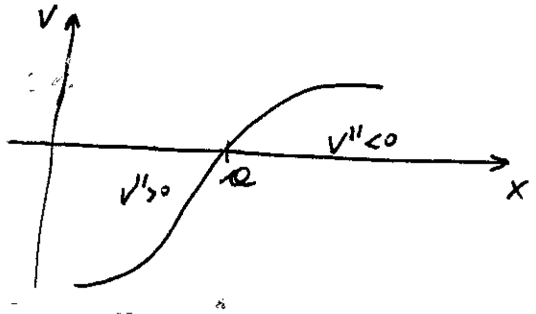
THIS IS CALLED "PROSPECT THEORY"

THEREFORE, IN THIS THEORY, U IS CONCAVE FUNCTION OF THE <u>CHANGE</u> OF MONEY (AND NOT OF MONEY)

8.6

WE DEFINE V, VALUE FUNCTION,

CONCAVE FOR $x > R$ AND CONVEX FOR $x < R$



WHERE R IS THE PERCEIVED WEALTH REFERENCE (USUALLY $\emptyset$)

SINCE MANY SYMMETRIC BETS $L_S = (-S, 50\% ; +S, 50\%)$

$S > 0$ ARE UNATTRACTIVE, AND THEY BECOME MORE UNATTRACTIVE THE LARGER S BECOMES, WE MAY SAY THAT

$$E(V(L_x)) < E(V(L_y)) \quad \forall x > y \quad \forall x > 0$$

$$\Downarrow$$

$$\frac{1}{2} V(x) + \frac{1}{2} V(-x) < \frac{1}{2} V(y) + \frac{1}{2} V(-y) \quad \forall x > y \quad \forall x > 0$$

$$\Downarrow$$

$$V(x) + V(-x) < V(y) + V(-y) \quad \forall x > y \quad \forall x > 0$$

TAKE $y = x + \Delta$ $\quad (\Delta < 0 \Rightarrow$ IT IS $y < x) \quad \forall x > 0$

$$V(x) - V(x+\Delta) < V(-x-\Delta) - V(-x) \quad \forall x > 0 \ \forall \Delta < 0$$

$$V(x+\Delta) - V(x) > V(-x) - V(-x-\Delta) \quad \forall x > 0 \ \forall \Delta < 0$$

$$\frac{V(x+\Delta) - V(x)}{\Delta} \underset{\Delta \text{ IS NEGATIVE}}{<} -\frac{V(-x-\Delta) - V(-x)}{\Delta} = \frac{V(-x-\Delta) - V(-x)}{-\Delta} = \frac{V(-x+(-\Delta)) - V(-x)}{(-\Delta)}$$

$\Delta \to 0$
$\Downarrow$
$-\Delta \to 0$

$$\Rightarrow \boxed{V'(x) < V'(-x) \quad \forall x > 0}$$

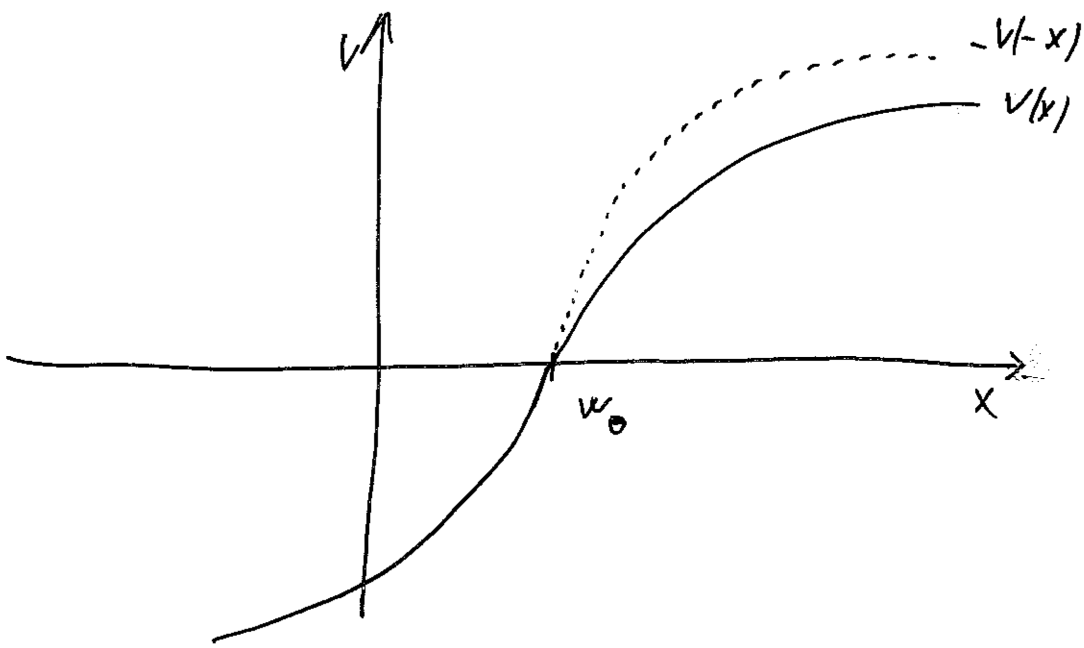
8.7) SUMMING UP, THE FEATURES OF A VALUE FUNCTION ARE

1) $V(w_0) = 0$ [NOT IMPORTANT]

2) V IS STRICTLY INCREASING. WHEN V IS DIFFERENTIABLE, THIS BECOMES $V'(x) > 0$

3) V IS NON-STRICTLY CONVEX FOR $X < w_0$ AND NON-STRICTLY CONCAVE FOR $X > w_0$.
WHEN V IS TWICE-DIFFERENTIABLE, $V''(x) = \begin{cases} \geq 0 & X < w_0 \\ \leq 0 & X > w_0 \end{cases}$

4) V IS STEEPER FOR $X < w_0$ THAN FOR $X > w_0$
WHEN V IS DIFFERENTIABLE, $V'(X + w_0) < V'(-X + w_0)$ $\underline{\underline{\forall x > 0}}$



8.8)

$$V(x) = \begin{cases} a\sqrt{-x} & x < 0 \\ b\sqrt{x} & x \geq 0 \end{cases}$$

FOR WHICH a, b IS $V(x)$ A VALUE FUNCTION?

$$V(0) = b\sqrt{0} = 0$$

$$V'(x) = \begin{cases} \dfrac{-a}{2\sqrt{-x}} & x < 0 \\ \dfrac{b}{2\sqrt{x}} & x > 0 \end{cases}$$

$$V'(x) > 0 \Rightarrow a < 0 \quad \text{AND} \quad b > 0$$

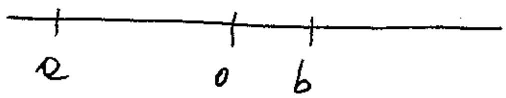
$$V''(x) = \begin{cases} -\dfrac{a}{4\sqrt{(-x)^3}} & x < 0 & \text{POS} \\ \dfrac{-b}{4\sqrt{x^3}} & x > 0 & \text{NEG} \end{cases}$$

OK

$$V'(x) < V'(-x) \quad \forall x > 0 \Rightarrow \frac{+b}{2\sqrt{+x}} < \frac{-a}{2\sqrt{-(-x)}} \quad \forall x > 0 \Rightarrow$$

$$\Rightarrow b < -a$$

THEREFORE $\boxed{a < 0}$ $\boxed{b > 0}$ $\boxed{b < -a}$



$\square$

$\boxed{8.9}$

$$V(x) = \begin{cases} a\sqrt{-x} + 2x & x < 0 \\ b\sqrt{x} + x & x \geq 0 \end{cases}$$

HW

FOR WHICH a, b IS $V(x)$ A VALUE FUNCTION?

$$V(0) = b\sqrt{0} + 0 = 0$$

$$V'(x) = \begin{cases} \dfrac{-a}{2\sqrt{-x}} + 2 & x < 0 \\ \dfrac{b}{2\sqrt{x}} + 1 & x > 0 \end{cases}$$

$$\frac{-a}{2\sqrt{-x}} + 2 > 0 \quad \forall x < 0$$

$$-a + 4\sqrt{-x} > 0 \quad \forall x < 0$$

$a \geq 0$

$4\sqrt{-x} > a$

$16(-x) > a^2 \quad \forall x < 0$

IF $a = 0$ OK

IF $a < 0$ IMPOSSIBLE

$a < 0$

IT WORKS FOR EVERY $a < 0$ AND EVERY $x < 0$

$$a \leq 0$$

$$\frac{b}{2\sqrt{x}} + 1 > 0 \quad \forall x > 0$$

$$b + 2\sqrt{x} > 0 \quad \forall x > 0$$

$b > 0$

IT WORKS FOR EVERY $b > 0$ AND $x > 0$

$b \leq 0$

$2\sqrt{x} > -b$

$4x > (-b)^2$

IF $b = 0$ OK

IF $b < 0$ IMPOSSIBLE

$$b \geq 0$$

$$V''(x) = \begin{cases} \dfrac{-a}{4(\sqrt{-x})^3} & x < 0 \quad \text{POS IF } \boxed{a < 0} \\ \dfrac{-b}{4\sqrt{x^3}} & x > 0 \quad \text{NEG IF } \boxed{b > 0} \end{cases}$$

$$V'(x) < V'(-x) \quad \forall x > 0 \Rightarrow \frac{b}{2\sqrt{x}} + 1 < -\frac{a}{2\sqrt{-(-x)}} + 2 \quad \forall x > 0 \overset{\text{SINCE } 2\sqrt{x} > 0}{\Rightarrow}$$

$$b < -a + 2\sqrt{x} \Rightarrow (b + a) < 2\sqrt{x} \quad \forall x > 0 \Rightarrow b + a \leq 0 \Rightarrow \boxed{b \leq -a} \quad \square$$

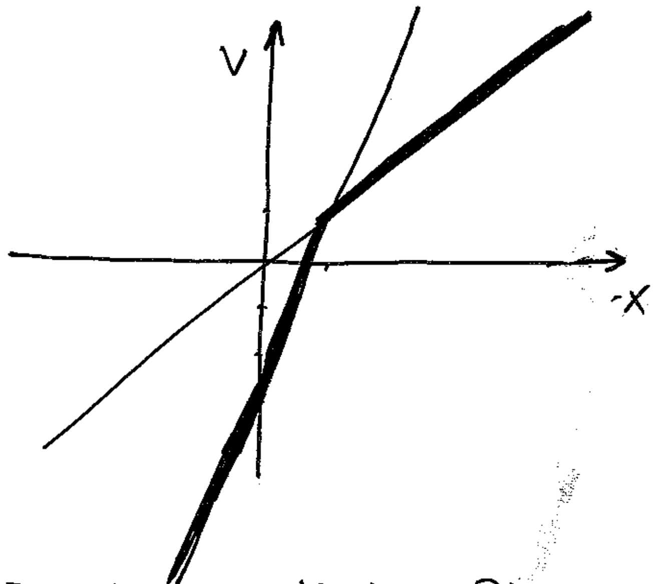
8.10

$V(x) = \min(4x-3, x)$

IS THIS A VALUE FUNCTION?

WHAT IS ITS REFERENCE POINT w_0?

THE GRAPH IS



FROM THE GRAPH WE IMMEDIATELY SEE THAT

- IT IS INCREASING
- IT IS NON DIFFERENTIABLE IN $x=1$
- IT IS LINEAR EVERYWHERE EXCEPT IN $x=1$
- IT IS STEEPER FOR $x<1$ THAN FOR $x>1$

LOOKING AT THE LAST CONSIDERATION, WHICH IS SIMILAR TO POINT 4, WE FIX $w_0 = 1$

LET'S CHECK

1) $V(1) = 1$ NOT SATISFIED, BUT IT IS NOT IMPORTANT

2) V IS INCREASING <u>OK</u>

3) V'' $x<1$ 0 NON STRICTLY CONVEX; $x>1$ 0 NON STRICTLY CONCAVE <u>OK</u>

4) STEEPER FOR $x<1$ (DERIVATIVE IS 4) THAN FOR $x>1$ (DERIVATIVE IS 1) <u>OK</u>

$\square$

8.11

IF WE WANT TO SATISFY 1, WE CAN PUT $W_0 = \frac{3}{4}$.

IN THIS WAY $V(W_0) = 0$

BUT CONDITION 4 IS NOT SATISFIED, SINCE STEEPNESS FOR $X < \frac{3}{4}$ IS ALWAYS 4 BUT STEEPNESS FOR $X > \frac{3}{4}$ IS SOMETIMES 4, SOMETIMES 1 AND SOMETIMES IT DOES NOT EXIST (IN $X=1$)

SO STEEPNESS FOR $X < W_0$ IS NOT <u>ALWAYS</u> STRICTLY LARGER THAN STEEPNESS FOR $X > W_0$.

8.12) is $V(x) = \int_0^x f(t)\,dt \quad f(t) = e^{-|t| \max(\alpha t, -\beta t)}$

$\alpha, \beta > 0$

A VALUE FUNCTION ? $W_0 = 0$

$$f(t) = \begin{cases} t<0 & e^{-t^2\beta} \\ t=0 & 1 \\ t>0 & e^{-t^2\alpha} \end{cases}$$

1) $V'(x) = \frac{d}{dx} \int_0^x f(t)\,dt = f(x)$ $\quad V'(x) > 0 \ \forall x$ OK

2) $V(0) = \int_0^0 f(t)\,dt = 0$ OK

3) $$V''(x) = \frac{d}{dx}\left(f(x)\right) = \begin{cases} -2x\beta\, e^{-x^2\beta} & x<0 \rightarrow \text{POS} \\ \text{DOES NOT EXIST} & x=0 \\ -2x\alpha\, e^{-x^2\alpha} & x>0 \rightarrow \text{NEG} \end{cases}$$ OK

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4) $$V'(x) = \begin{cases} e^{-x^2\beta} & x<0 \\ e^{-x^2\alpha} & x>0 \end{cases}$$

$V'(x)$ FOR A POSITIVE x is $e^{-x^2\alpha}$

$V'(-x)$ FOR A POSITIVE x is $e^{-(-x)^2\beta} = e^{-x^2\beta}$

$e^{-x^2\alpha}$ MUST BE SMALLER THAN $e^{-x^2\beta}$ AND THIS IS TRUE ONLY WHEN $\alpha > \beta$

WHEN $\alpha > \beta$, $V(x)$ IS A VALUE FUNCTION

D

8.13

WHY IS A VALUE FUNCTION STEEPEST AT ITS REFERENCE POINT?

SUPPOSE V IS TWICE-DIFFERENTIABLE

① $V(0)=0$
② $V'(x)>0 \quad \forall x$
③ $V''(x)\geq 0 \quad \forall x<0 \qquad V''(x)\leq 0 \quad \forall x>0$
④ $V'(x)< V'(-x) \quad \forall x>0$

STEEPEST MEANS THAT ITS FIRST DERIVATIVE HAS THE LARGEST POSSIBLE VALUE. THIS VALUE IS ALWAYS POSITIVE BECAUSE OF ①.

→ FOR $X<0$, $V''\geq 0 \Rightarrow (V')'\geq 0 \Rightarrow$ THE DERIVATIVE OF THE FIRST DERIVATIVE IS POSITIVE $\Rightarrow$ THE FIRST DERIVATIVE IS INCREASING

→ FOR $X>0$, $V''\leq 0 \Rightarrow (V')'\leq 0 \Rightarrow$ THE FIRST DERIVATIVE IS DECREASING

$\Downarrow$

$X=0$ IS THE POINT WHERE V' STOPS INCREASING AND START DECREASING, THEREFORE IT IS A LOCAL MAXIMUM

SINCE $V''(x)\geq 0$ FOR EVERY $x<0 \Rightarrow V'$ IS ALWAYS SMALLER ON THE LEFT OF 0

$V''(x)\leq 0$ FOR EVERY $x>0 \Rightarrow V'$ IS ALWAYS SMALLER ON THE RIGHT OF 0

$\Downarrow$

IT IS A GLOBAL MAXIMUM

D

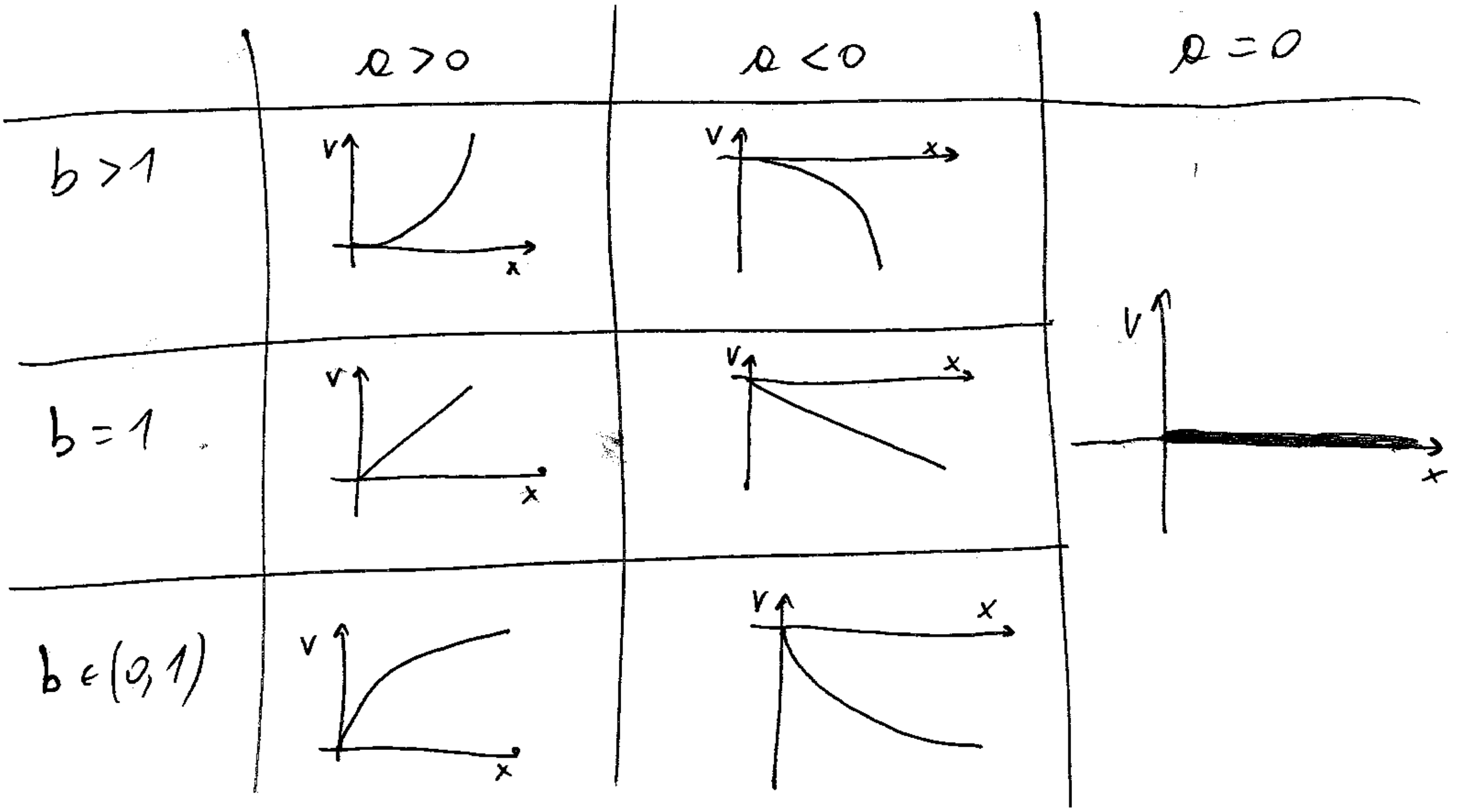
8.14 | HW GRAPHICALLY

FOR WHICH a, b, c, d is $V(x)$ A CONTINOUS VALUE FUNCTION?

$$V(x) = \begin{cases} a x^b & x \geq 0 \\ c(-x)^d & x < 0 \end{cases}$$

FIRST WE EXCLUDE THE VALUES $b \leq 0$ BECAUSE THEY CREATE DEFINITION PROBLEMS WHEN $x=0$.

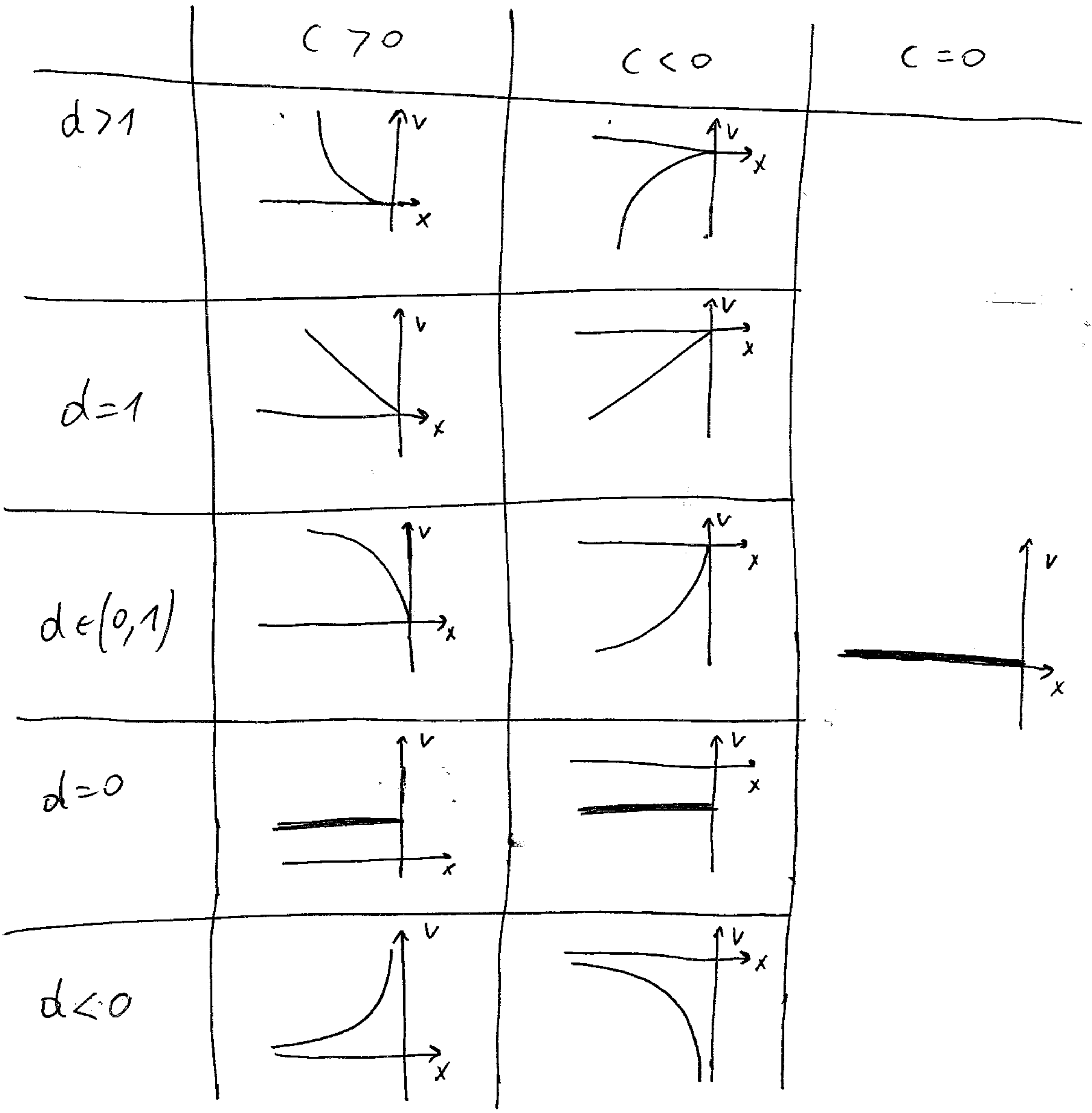
THEN, LET'S LOOK AT $a x^b$, FOR $\boxed{b>0}$



THE ONLY POSSIBLE VALUES TO HAVE $V(x)$ INCREASING AND CONCAVE FOR $X \geq 0$ ARE THEREFORE $b \in [0,1]$ AND $a > 0$

8.15

LET'S LOOK AT $c(-x)^d$



THE ONLY POSSIBLE VALUES FOR $V(x)$ TO BE INCREASING AND CONVEX FOR $X<0$ ARE $c<0$ AND $d \in [0,1]$.
THE CASE $d<0$ AND $c>0$ IS EXCLUDED BECAUSE $V(x)$ WILL NOT BE CONTINOUS.

THESE CONDITIONS HOWEVER DO NOT CONSIDER (4)

8.16

HOWEVER THE LAST CONDITION MUST BE CHECKED ANALITICALLY

$b \in (0,1]$ $a \in (0,+\infty)$ $c \in (-\infty,0)$ $d \in (0,1]$

$V'(x) < V'(-x) \quad \forall x > 0$

$a b x^{b-1} < cd(-(-x))^{d-1} \cdot (-1) \quad \forall x>0$

$x^{b-1-d+1} < -\frac{cd}{ab}$ $\quad x^{b-d} < -\frac{cd}{ab}$ $\quad -\frac{cd}{ab}$ IS POSITIVE

THIS INEQUALITY $x^{(b-d)} < -\frac{cd}{ab}$ MUST HOLD TRUE $\forall x>0$

IF $b-d>0$ IT WILL NOT HOLD TRUE FOR VERY LARGE VALUES OF x

IF $b-d<0$ IT WILL NOT HOLD TRUE FOR VERY SMALL VALUES OF x

IF $\boxed{b=d}$ $\quad 1 < -\frac{cd}{ab} \Rightarrow ab < -cd \Rightarrow \boxed{a<-c}$

$\Downarrow$

$a \in (0,-c)$ $b \in (0,1]$ $c \in (-\infty,0)$ $d=b$