Advanced insurance.

Consider a risky situation where the loss (indicated with positive values $x \geq 0$) is a continuous r.v. L with density f(x). We assume that $\mathrm{E(L)} > 0$, which means that L is not always 0.

We define I(x) what the insurance company pays to the customer in case of loss x. Clearly $0 \leq \mathrm{I}(x) \leq x$ since no costumer would accept negative reimbursements (therefore having to pay money to the insurance company instead of receiving) and no insurance company reimburse more that the effective loss.

I(x) is

- full coverage: $\mathrm{I}(x) = x$
- proportional: $\mathrm{I}(x) = kx$, where $k \in (0;1)$
- limited coverage: $\mathrm{I}(x) = \begin{cases} x \leq m & x \\ x > m & m \end{cases}$
- stop loss: $\mathrm{I}(x) = \begin{cases} x \leq d & 0 \\ x > d & x-d \end{cases}$, where d is the deductible.

We define the **expected claim** as the expected value of the money received by the costumer. It is sometimes called **expected loss** since for the insurance company it represents a loss. Its definition is $\mathrm{E}(\mathrm{I}(x)) = \int_0^{+\infty} \mathrm{I}(s) f(s) \mathrm{ds}$. On the expected claim we have the restriction that is must be positive or zero since both functions are positive or zero. Moreover, $\int_0^{+\infty} \mathrm{I}(s) f(s) \mathrm{ds} \leq \int_0^{+\infty} s f(s) \mathrm{ds} = \mathrm{E(L)}$.

For the stop-loss insurance the expected claim becomes

$$\mathrm{E}(\mathrm{I}(x)) = \int_0^{d} 0 f(s) \mathrm{ds} + \int_d^{+\infty} (s-d) f(s) \mathrm{ds} = \int_d^{+\infty} (s-d) f(s) \mathrm{ds}$$

9.2

$R(r) = E(I_r(L))$ DEFINITION

$$R(r) = \int_r^{+\infty} (s-r) f(s) ds = \int_0^{+\infty} (s-r) f(s) ds - \int_0^r (s-r) f(s) ds =$$

$$= \int_0^{+\infty} s f(s) ds - \int_0^r s f(s) ds - r \int_0^{+\infty} f(s) ds + r \int_0^r f(s) ds =$$

$$= \left[E(L) - \int_0^r s f(s) ds \right] - \left[r - r \int_0^r f(s) ds \right] = R_1(r) - R_2(r)$$

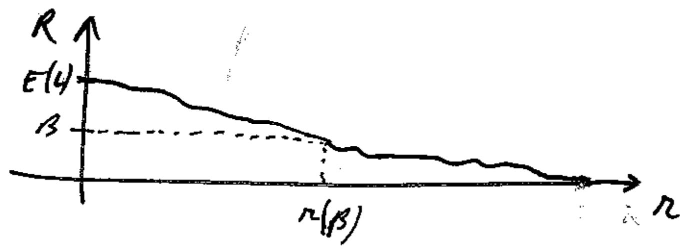
$$R'(r) = -r f(r) - 1 + \int_0^r f(s) ds + r \cdot f(r) = \int_0^r f(s) ds - 1$$

SINCE $\int_0^r f(s) < 1 \Rightarrow R'(r) < 0 \Rightarrow R$ STRICTLY DECREASING

$R(0) = \int_0^{+\infty} (s-0) f(s) ds = E(L) > 0$ SINCE THERE IS A LOSS IN THIS LOTTERY

$$\lim_{r \to +\infty} R(r) = \lim_{r \to +\infty} \int_r^{+\infty} (s-r) f(s) ds \underset{\substack{\text{WE DO NOT} \\ \text{SHOW THE} \\ \text{PROOF}}}{=\!=\!=} 0$$

R IS DECREASING, STARTS FROM $E(L)$ AND GOES TO $\emptyset$



THEREFORE FOR ANY $\beta \in (0, E(L))$ THERE IS A CORRESPONDING r, CALLED $r(\beta)$, WHICH

$$R(r(\beta)) = \beta$$

AND THIS $r(\beta)$ IS UNIQUE.

9.3) EXAMPLE: $L \sim \lambda e^{-\lambda x}$ STOP-LOSS INSURANCE

FIND $r(\beta)$

NOTE THAT $E(L) = \frac{1}{\lambda}$

$$R(r) = E(I_r(L)) = \int_r^{+\infty} (s-r)\lambda e^{-\lambda s}\, ds = \text{BY PARTS}$$

$$= \left[(s-r)\left(-\frac{1}{\lambda}\right)\lambda e^{-\lambda s}\right]_r^{+\infty} - \int_r^{+\infty}\left(-\frac{1}{\lambda}\right)\lambda e^{-\lambda s}\, ds =$$

$$= 0 + \frac{1}{\lambda}(r-r)\lambda e^{-\lambda r} + \left[-\frac{e^{-\lambda s}}{\lambda}\right]_r^{+\infty} =$$

$$= 0 + 0 - \frac{1}{\lambda}\left[e^{-\lambda s}\right]_r^{+\infty} = 0 + \frac{1}{\lambda} e^{-\lambda r}$$

$$R(r) = \frac{1}{\lambda} e^{-\lambda r}$$

$$R(r(\beta)) = \beta \qquad \frac{1}{\lambda} e^{-\lambda r(\beta)} = \beta \qquad e^{-\lambda r(\beta)} = \beta\lambda$$

$$-\lambda r(\beta) = \ln \beta\lambda \qquad \boxed{r(\beta) = -\frac{1}{\lambda}\ln \beta\lambda}$$

FOR AN EXPECTED CLAIM OF β THE DEDUCTABLE MUST BE $-\frac{1}{\lambda}\ln\beta\lambda$

NOTE THAT $\beta \leq \frac{1}{\lambda}$ SINCE $E(L) = \frac{1}{\lambda}$ AND R MUST BE $\leq E(L)$

$\Downarrow$

$\ln\beta\lambda \leq 0 \Rightarrow r(\beta) \geq 0$

9.4 EXERCISE

STOP-LOSS INSURANCE WITH DEDUCTIBLE d

LOSS IS UNIFORMLY DISTRIBUTED OVER $[0, 3d]$

WHAT IS EXPECTED CLAIM ?

EXPECTED CLAIM $= E(I_d(L)) = \int_0^{+\infty} I_d(s)\, f(s)\, ds = \int_0^d 0 \cdot f(s)\, ds + \int_d^{3d} (s-d) \cdot f(s)\, ds$

$$f(s) = \begin{cases} \frac{1}{3d} & s \in [0, 3d] \\ 0 & s \notin [0, 3d] \end{cases} \qquad I_d(s) = \begin{cases} 0 & s \leq d \\ s-d & s > d \end{cases}$$

$$E(I_d(L)) = \int_d^{3d} (s-d) \cdot \frac{1}{3d}\, ds = \left[\frac{(s-d)^2}{6d}\right]_d^{3d} = \frac{(3d-d)^2}{6d} - \frac{(d-d)^2}{6d} = \frac{4d^2}{6d} = \frac{2}{3} d$$

$\square$

9.5

EXERCISE

HW STOP-LOSS INSURANCE

LOSS is $U(0;5)$ $E(I_d(L))=1$

FIND d

$$1 = E(I_d(L)) = \int_0^d 0 \cdot f(s)\,ds + \int_d^{+\infty} (s-d) f(s)\,ds =$$

$$f(s) = \begin{cases} \frac{1}{5} & s \in [0,5] \\ 0 & s \notin [0,5] \end{cases} \qquad I_d(s) = \begin{cases} 0 & s \leq d \\ s-d & s > d \end{cases}$$

$$= \int_d^5 (s-d)\frac{1}{5}\,ds = \left[\frac{(s-d)^2}{10}\right]_d^5 = \frac{(5-d)^2}{10} - \frac{(d-d)^2}{10} = \frac{(5-d)^2}{10}$$

$$\frac{(5-d)^2}{10} = 1 \qquad (5-d)^2 = 10 \qquad 5-d = \pm\sqrt{10}$$

$$d = 5 \mp \sqrt{10} = \begin{cases} 5+\sqrt{10} \\ 5-\sqrt{10} \end{cases}$$

$5+\sqrt{10}$ → THIS VALUE DOES NOT MAKE SENSE BECAUSE d MAY NOT BE LARGER THAN THE MAXIMUM LOSS!

$$\boxed{d = 5-\sqrt{10}}$$

D

9.6 OPTIMAL INSURANCE THEOREM

DECISION MAKER WITH

1) RISK AVERSE WITH U TWICE-DIFFERENTIABLE
2) L IS A LOSS WITH $f(x)$ DENSITY FUNCTION $x \in [0, +\infty)$
3) READY TO SPEND P ON AN INSURANCE

4) AMONG ALL CONTRACTS $I(x)$ WITH $0 \leq I(x) \leq x$
WITH $E[I(L)] = \beta$ EXPECTED CLAIM

HIS EXPECTED UTILITY IS MAXIMIZED CHOOSING A
STOP-LOSS CONTRACT WITH
DEDUCTIBLE r SUCH THAT $\beta = \int_r^{+\infty} (s-r) f(s)\, ds$

NOTES:

* r, DOES NOT DEPEND ON U PROVIDED THAT $U'' \leq 0$

* WHAT DOES MAXIMIZING EXPECTED UTILITY MEAN?
IN THIS CASE IT MEANS MAXIMIZING THE AMOUNT
OF MONEY RECEIVED FROM THE INSURANCE COMPANY

$$E(U(I(L))) = \int_0^{+\infty} U(I(s) + w_0) \cdot f(s)\, ds$$

WITH THE BOUND THAT $\int_0^{+\infty} I(s) \cdot f(s) = \beta$

WHICH, FOR A STOP LOSS, MEANS $\int_r^{+\infty} (s-r) f(s) = \beta$

NOTE: β IS ALSO CALLED EXPECTED LOSS

9.7 IMAGINE A RISKY SITUATION WITH $R(0;7)$

CONSIDER ALL INSURANCES WITH EXPECTED CLAIM EQUAL TO 0,2 . SUPPOSE $U(x)$ TWICE DIFFERENTIABLE $U''<0$. SUPPOSE ALL INSURANCES REQUIRE THE SAME PREMIUM. FIND THE BEST.

USING OPTIMAL INSURANCE THEOREM, SINCE U IS TWICE DIFF, AMONG ALL CONTRACTS WITH $0 \le I(x) \le x$ WITH $E(I(x)) = 0,2$, THE EXPECTED UTILITY IS MAXIMIZED CHOOSING

$$0,2 = \int_d^{+\infty} (s-d)\, f(s)\, ds = \int_d^7 (s-d) \cdot \frac{1}{7}\, ds = \left[\frac{(s-d)^2}{14}\right]_d^7 =$$

$$= \frac{(7-d)^2}{14} - 0 = \frac{49-14d+d^2}{14}$$

$$2,8 = 49 - 14d + d^2 \qquad d^2 - 14d + 46,2 = 0$$

$$d_{1,2} = \frac{7 \pm \sqrt{49-46,2}}{1} = 7 \pm \sqrt{2,8} = \begin{cases} 8,67 \rightarrow \text{NOT POSSIBLE SINCE LOSS IS } R(0;7) \\ 5,33 \end{cases}$$

$d = 5,33$ IS THE BEST DEDUCTIBLE

$\boxed{9.8}$

HW

U TWICE-DIFFERENTIABLE $\quad U'' \leq 0$

$L \sim \lambda e^{-\lambda x} \quad \lambda > 0$

EXPECTED CLAIM IS $\frac{1}{2\lambda}$

WHAT IS THE BEST POSSIBLE INSURANCE CONTRACT.

ACCORDING TO THE OPTIMAL INSURANCE THEOREM

<u>IF</u> AGENT IS RISK-AVERSE THE OPTIMAL CONTRACT IS A STOP-LOSS WITH DEDUCTIBLE d

$$\frac{1}{2\lambda} = \int_d^{+\infty} (s-d)\, \lambda e^{-\lambda s}\, ds$$

$$\int_d^{+\infty} s \lambda e^{-\lambda s}\, ds - d \int_d^{+\infty} \lambda e^{-\lambda s}\, ds = \left[s \cdot (-e^{-\lambda s}) \right]_d^{+\infty} - \int_d^{+\infty} 1 \cdot (-e^{-\lambda s})\, ds - d\left[-e^{-\lambda s} \right]_d^{+\infty} =$$

$$= 0 + \cancel{d e^{-\lambda d}} + \left[-\frac{e^{-\lambda s}}{\lambda} \right]_d^{+\infty} + d\left(0 \cancel{- e^{-\lambda d}}\right) = +\frac{e^{-\lambda d}}{\lambda}$$

$$\frac{1}{2\lambda} = \frac{e^{-\lambda d}}{\lambda} \qquad \ln \frac{1}{2} = -\lambda d \qquad d = -\frac{1}{\lambda} \ln \frac{1}{2} = \frac{1}{\lambda} \ln 2$$

$\square$

9.9 $I(x) = \sqrt{x}$ DOES THIS CORRESPONDS TO A FEASIBLE <u>INSURANCE</u> CONTRACT ?

ON $I(x)$ WE HAVE THE RESTRICTIONS

$$0 \le I(x) \le x$$

$0 \le I(x)$ IS TRUE FOR $\sqrt{x}$

<u>BUT</u> $\sqrt{x} \le x$ IS <u>NOT</u> TRUE (FOR EXAMPLE $x = 0{,}5$ DOES NOT WORK)

$\square$