10.1 INVESTOR AND ASSETS

$$\text{RATE OF RETURN} = \frac{\text{WEALTH AT END} - \text{WEALTH AT BEGINNING}}{\text{WEALTH AT BEGINNING}}$$

$$\text{r.o.r.} \in [-1, +\infty)$$

- FOR RISK-FREE INVESTMENTS, r IS CALCULATED.
- FOR RISKY INVESTMENTS, SUCH AS STOCKS, r CANNOT BE CALCULATED BECAUSE "WEALTH AT END" DEPENDS ON UNKNOWN FUTURE DIVIDENDS AND EXPECIALLY ON HIGHLY-FLUCTUATING PRICE OF THE STOCK AT END.

STOCK PRICES HAVE A LOGNORMAL DISTRIBUTION $LN(\mu, \theta^2)$

P_A INITIAL PRICE $\quad P_B$ FUTURE PRISE $\sim LN(\mu, \theta^2)$

Σ QUANTITY OF STOCKS

$$\text{r.o.r.} = \frac{P_B \Sigma - P_A \Sigma}{P_A \Sigma} = \frac{P_B - P_A}{P_A} = \frac{P_B}{P_A} - 1 \qquad (\text{r.o.r.} + 1) P_A = P_B$$

$$\Downarrow$$

$\text{r.o.r.} + 1 \sim LN(\mu, \theta^2)$ WITH A P SCALING PARAMETER

10.2

CONSIDER AN INITIAL WEALTH W
AND A PORTFOLIO DIVIDED IN
$W-X$ ON RISK-FREE ASSET WITH r RATE OF RETURN
X ON RISKY ASSET WITH ξ R.O.R. RANDOM VARIABLE

THE FINAL WEALTH IS

$$w_1 = (W-X)(1+r) + X\cdot(1+\xi) = W - \cancel{X} + Wr - Xr + \cancel{X} + X\xi =$$
$$= W(1+r) + X(\xi - r)$$

DEFINE $w_0 = W(1+r)$, THE FINAL WEALTH IF WE INVEST EVERYTHING IN A RISK-FREE ASSET, AND $\tilde{\xi} = \xi - r$, THE EXCESS RETURN.

$$w_1 = w_0 + X\tilde{\xi}$$

PROBLEM OF INVESTOR: CHOOSE X TO MAXIMIZE

$$V(X) = E\left(U\left(w_0 + X\tilde{\xi}\right)\right)$$

DISCRETE RANDOM VARIABLE

$$V'(X) = \frac{d}{dx} E\left(U\left(w_0 + X\tilde{\xi}\right)\right) = \frac{d}{dx}\left(\sum_{i=1}^{n} p_i \cdot U\left(w_0 + X\tilde{x}_i\right)\right) =$$
$$= \sum_{i=1}^{n} p_i U'\left(w_0 + X\tilde{x}_i\right)\cdot \tilde{x}_i = E\left(U'\left(w_0 + X\tilde{\xi}\right)\cdot\tilde{\xi}\right)$$

$V'(X^*) = 0$ TO FIND A STATIONARY POINT X^*

CONTINOUS RANDOM VARIABLE

$$V'(X) = \frac{d}{dX}\int_{-\infty}^{+\infty} \tilde{f}(s)\, U(w_0 + Xs)\, ds = \int_{-\infty}^{+\infty} \tilde{f}(s)\, U'(w_0 + Xs)\cdot s\, ds =$$
$$= E\left(U'\left(w_0 + X\tilde{\xi}\right)\cdot\tilde{\xi}\right)$$

$$V''(X) = \frac{d}{dX} V'(X) = E\left(U''\left(w_0 + X\tilde{\xi}\right)\cdot\tilde{\xi}^2\right)$$

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SUPPOSE U IS TWICE-DIFFERENTIABLE AND THAT THE INVESTOR IS RISK-AVERSE

$$\Downarrow$$

$$U''<0$$

$$\Downarrow$$

$$E\left(U''(w_0+X\tilde{\varrho})\,\tilde{\varrho}^2\right)<0$$

$$\Downarrow$$

$$V''<0$$

$$\Downarrow$$

X^*, A STATIONARY POINT (FOR WHICH $V'=0$), IS A MAXIMUM

MOREOVER, SINCE $V''<0 \Rightarrow V'$ IS DECREASING

$$\Downarrow$$

IF $V'(0)>0$ THE POINT X^* MUST BE ON THE RIGHT OF $\emptyset$ $\Downarrow$ $X^*>0$

IF $V'(0)=0$ X^* IS 0 $X^*=0$

IF $V'(0)<0$ THE POINT X^* MUST BE ON THE LEFT OF $\emptyset$ $\Downarrow$ $X^*<0$

$$\Downarrow$$

X^* HAS THE SAME SIGN OF $V'(0)$ (IF IT EXISTS)

$$V'(0)=E\left(U'(w_0)\,\tilde{\varrho}\right)=U'(w_0)\cdot E(\tilde{\varrho})$$

SINCE $U'>0 \Rightarrow$ X^* AND $E(\tilde{\varrho})$ HAVE THE SAME SIGN

THEREFORE THE <u>OPTIMAL</u> AMOUNT TO INVEST IN A RISKY ASSET IS NEGATIVE, ZERO, POSITIVE IF THE EXPECTED EXCESS RETURN IS NEGATIVE, ZERO, POSITIVE.

NOTE THAT IF $\tilde{\varrho}$ HAS ONLY NEGATIVE/POSITIVE OUTCOMES, FOR EXAMPLE $\varrho<r$ / $\varrho>r$, $E(U'\cdot\tilde{\varrho})$ IS ALWAYS NEGATIVE/POSITIVE, AND <u>NO OPTIMAL</u> X EXISTS!

10.4

EXAMPLE

$\xi + 1 \sim LN(\mu, \theta^2)$

IS THE VARIATION OF THE PRICE, 1 + RATE OF RETURN, OF A STOCK ASSET.
ξ IS THE R.O.R.

THE OPTIMAL X^* IS FOUND FROM

$$E\left(U'(w_0 + X^* \cdot (\xi - r)) \cdot (\xi - r)\right) = 0$$

HOWEVER, WITHOUT CALCULATING IT, WE KNOW THAT IT IS POSITIVE IF AND ONLY IF

$$E(\xi - r) > 0$$

$$\Updownarrow$$

$$E(\xi) > r \Leftrightarrow E(\xi + 1 - 1) > r \Leftrightarrow E(\xi + 1) - 1 > r \Leftrightarrow$$

$$\Leftrightarrow \exp\left(\mu + \frac{\theta^2}{2}\right) > r + 1 \Leftrightarrow \mu + \frac{\theta^2}{2} > \ln(r+1)$$

IF $\mu + \frac{\theta^2}{2} > \ln(r+1)$ WE HAVE A POSITIVE OPTIMUM X^*

10.5

A RISK-AVERSE AGENT WITH U TWICE-DIFFERENTIABLE HAS A RISK-FREE ASSETS WITH 4 % RETURN AND A RISKY ASSETS WITH A RETURN $\xi \sim \lambda e^{-\lambda x}$ (NOT A RATE OF RETURN).

FIND THE VALUES OF λ FOR WHICH THE AGENT WOULD INVEST IN THE RISKY ASSETS

$r = 1.04$ R.O.R RISKY $\sim 1 + \xi$ WITH $\xi \sim \lambda e^{-\lambda x}$

IN ORDER TO FIND THE BEST X^*, THE OPTIMAL AMOUNT TO INVEST IN THE RISKY ASSETS, THE AGENT SHOULD MAXIMIZE

$$V(X) = E\left(U\left(w_0 + X\left(1+\xi - r\right)\right)\right)$$

BUT WE KNOW IN ADVANCE THAT X^* HAS THE SAME SIGN AS $V'(0)$ AND $E\left(1+\xi-r\right)$

$$E(1+\xi) = 1 + E(\xi) = 1 + \frac{1}{\lambda} \qquad E(1+\xi-r) = \frac{1}{\lambda} - 0.04$$

THIS QUANTITY IS POSITIVE WHEN $\frac{1}{\lambda} > 0.04$, $\lambda < \frac{1}{0.04}$, $\lambda < 25$

FOR $\lambda < 2.5$ $\quad E(1+\xi-r) > 0 \Rightarrow X^* > 0 \Rightarrow$

THE AGENT WILL INVEST IN THE RISKY ASSETS

$\square$

10.5

INVESTOR'S PROBLEM WITH BENCHMARK

$$W - X \quad + \quad X \quad = \quad W$$

$\downarrow r$ $\qquad \downarrow \xi$ $\qquad$ WANTS AT LEAST $\downarrow$ W_B

$$W_1 = W_0 - Xr + X\xi$$

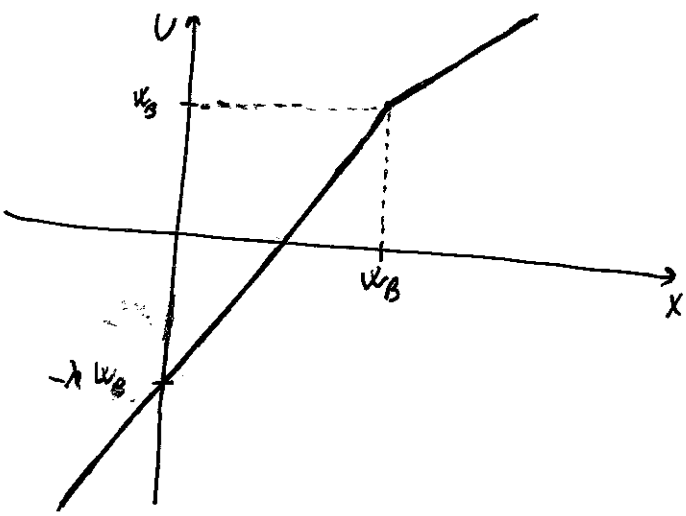
$r \in (0, +\infty)$

$\xi \in [-1, +\infty)$

W_B IS THE BENCHMARK LEVEL, THE MINIMAL AMOUNT OF MONEY I EXPECT FROM MY ASSET ALLOCATION

THE UTILITY FUNCTION IS THEREFORE BUILT IN SUCH A WAY TO PENALIZE RESULTS SMALLER THAN W_B

$$U(x) = x - \lambda \cdot \max(W_B - x, 0) \qquad \lambda > 0$$



IT IS NOT DIFFERENTIABLE IN W_B

IT IS CONCAVE

$\Downarrow$

AGENT IS RISK-AVERSE

<u>WE MUST MAXIMIZE</u> $E(U(W_1)) = E(W_1 - \lambda \max(W_B - W_1, 0)) =$

$$= E(W_1) - \lambda E(\max(W_B - W_1, 0)) = E(W_0 + X(r - \xi)) - \lambda E(\max(W_B - W_1, 0)) =$$

$$= E(W_0 + X(-r + \xi)) - \lambda E(\max(W_B - W_0 + X(r - \xi), 0)) = V(X)$$

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λ IS THE RISK-AVERSION PARAMETER

$-\lambda E(\max(W_B - W_1), 0)$ IS THE EXPECTED SHORTFALL OR DOWNSIDE RISK

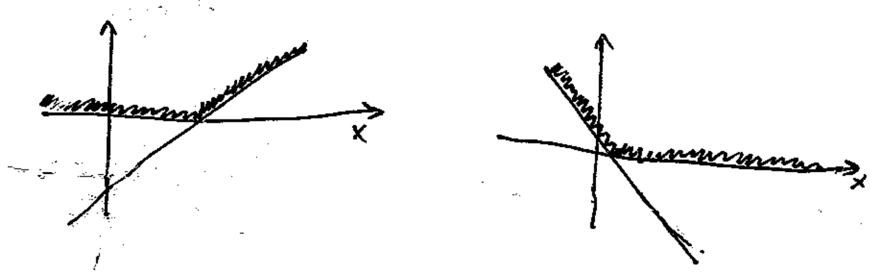
AND IT IS 0 IF $W_1 \geq W_B$ WHILE NEGATIVE IF $W_1 < W_B$

LET'S STUDY THE CONCAVITY OF $V(X)$:

$$E(W_0 + X(r - \xi)) = W_0 + X(r - E(\xi)) \quad \text{IT IS LINEAR IN } X$$

$$E(\max(W_B - W_0 - X(\xi - r), 0)) = \sum_{i=1}^{m} \max(W_B - W_0 - X(x_i - r), 0) \cdot p_i$$

AND $W_B - W_0 - X(x_i - r)$ IS LINEAR IN X. THE MAXIMUM OF ZERO AND A LINEAR FUNCTION IS



mmm = MAX

ALWAYS CONVEX. A LINEAR COMBINATION (THROUGH p_i) OF CONVEX FUNCTIONS IS STILL CONVEX. MULTIPLIED BY $-\lambda$, IT BECOMES CONCAVE.

$\Downarrow$

$$-\lambda E(\max(W_B - W_0 - X(\xi - r), 0)) \text{ IS CONCAVE}$$

$\Downarrow$

$V(X)$ IS CONCAVE, AS SUM OF LINEAR AND CONCAVE FUNCTIONS

$\Downarrow$

IF WE FIND A POINT X^* SUCH THAT $V'(X^*) = 0$

$\Downarrow$

X^* IS A MAXIMUM

10.7

WE HAVE HERE THREE POSSIBILITIES

$W_B > W_0$ IN THIS CASE INVESTING EVERYTHING IN THE RISK-FREE ASSET DOES NOT LET US REACH THE BENCHMARK. WE MUST HAVE $X > 0$ TO BE ABLE TO HOPE TO REACH W_B.

$W_B = W_0$ INVESTING EVERYTHING IN RISK-FREE ASSET, $X=0$, WILL LET US ARRIVE FOR SURE EXACTLY AT THE BENCHMARK LEVEL.

$W_B < W_0$ INVESTING EVERYTHING IN RISK-FREE ASSET WILL LET US OVERSHOOT THE BENCHMARK LEVEL! WE CAN TRY TO INVEST A PART $X > 0$ IN A RISKY ASSET. DEPENDING ON HOW MUCH, OUR PROBABILITY TO REACH AT LEAST W_B MAY STAY 100% OR DROP TO LOWER VALUES

THIS IS THE CASE WE WILL INVESTIGATE, BECAUSE HERE THE AGENT MUST CHOOSE BETWEEN INVESTING IN RISKY ASSETS OR NOT.

THEOREM: IF $\frac{E(\xi) - r}{\lambda} < R$ WHERE $R = \int_{-1}^{r} (r-s) f_\xi(s)\, ds$

THEN X^* EXISTS AND IS UNIQUE, SUCH THAT

$$V'(X^*) = E(\xi) - r - \lambda T(X^*) = 0$$

WHERE $T(X) = \int_{-1}^{r - \frac{W_0 - W_B}{X}} (r-s) f_\xi(s)\, ds$

If we know the value of r and the distribution of ξ and if the inequality of the theorem holds, now we can notice that given λ, thanks to the theorem, we can find the value of X^* from $E(\xi)-r-\lambda T(X^*)=0$ and we can define $x^* = r - \frac{W_0 - W_B}{X^*}$.
This quantity x^* has a specific meaning in economics: it represents the minimum rate of return for the risky asset, this means the minimum outcome for random variable ξ, for which we manage to reach the benchmark. In fact, if ξ has an outcome (at ending time) smaller than $r - \frac{W_0 - W_B}{X^*}$, then we have $W_1 = W_0 - X^* r + X^* \xi < W_0 - X^* r + X^* r - X^* \frac{W_0 - W_B}{X^*} = W_B$. For values of ξ larger than $r - \frac{W_0 - W_B}{X^*}$, we have $W_1 > W_B$.
Now we can find the probability of not reaching the benchmark $\gamma = P(\xi < x^*) = F_\xi(x^*)$.
So, starting from λ we can find all the quantities X^*, x^* and γ.

But we can also do the other way around, which is the procedure that is usually taken when facing a real problem: fixing a probability γ that is the maximum risk we want to take not to reach the benchmark, we can find x^* inverting the cumulative distribution function (which can be inverted when strictly increasing) $x^* = F^{-1}_\xi(\gamma)$. From here we can find $X^* = \frac{W_0 - W_B}{r - x^*}$. Using the just found value of X^*, we can find λ from equation $E(\xi) - r - \lambda T(X^*) = 0$.

<u>Example</u>: let ξ be uniformly distributed from r-a to r+b, with obviously $r-a \geq -1$ and $a<b$. Find the condition for existence and uniqueness and express λ in function of γ and express γ in function of λ.

First we write the condition for existence and uniqueness of X^*, which is

$$\frac{E(\xi) - r}{\lambda} < \int_{-1}^{r} (r - s) f_\xi(s) ds = \int_{r-a}^{r} (r - s) \frac{1}{a+b} ds = \left[-\frac{(r - s)^2}{2(a+b)} \right]_{r-a}^{r} = \frac{a^2}{2(a+b)}$$

Which, introducing the value of $E(\xi) = r + \frac{b-a}{2}$, becomes

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CONDITION IS $\dfrac{r+\frac{b-a}{2}-r}{\lambda} < \dfrac{a^2}{2(a+b)}$

$$\frac{b-a}{2\lambda} < \frac{a^2}{2(a+b)} \qquad \boxed{b^2-a^2 < a^2\lambda}$$

IF THIS CONDITION HOLDS, X^* EXISTS. OTHERWISE WE DO NOT KNOW WHETHER IT EXISTS.

TO FIND IT, LET'S EVALUATE $T(X)$

$$T(X) = \int_{-1}^{r-\frac{W_0-W_B}{X}} (r-s) f_Y(s)\, ds = \int_{-1}^{r-a} (r-s)\cdot 0\, ds + \int_{r-a}^{r-\frac{W_0-W_B}{X}} (r-s)\cdot \frac{1}{a+b}\, ds =$$

$$= \left[-\frac{(r-s)^2}{2(a+b)}\right]_{r-a}^{r-\frac{W_0-W_B}{X}} = -\frac{\left(r-r+\frac{W_0-W_B}{X}\right)^2}{2(a+b)} + \frac{(r-r+a)^2}{2(a+b)} =$$

$$= -\frac{(W_0-W_B)^2}{2X^2(a+b)} + \frac{a^2}{2(a+b)} = \frac{1}{2(a+b)}\left[a^2 - \frac{(W_0-W_B)^2}{X^2}\right]$$

$\rightarrow$ NOTE THAT THIS FORMULA IS VALID IF $r-\frac{W_0-W_B}{X} > r-a \Rightarrow$

$\Rightarrow \frac{W_0-W_B}{a} < X$. OTHERWISE IF $X \leq \frac{W_0-W_B}{a}$, THEN $T(X)=0$

TO FIND X^* $\quad E(\xi) - r - \lambda T(X^*) = 0$

$$\cancel{r} + \frac{b-a}{2} - \cancel{r} - \lambda \frac{1}{2(a+b)}\left[a^2 - \frac{(W_0-W_B)^2}{X^2}\right] = 0$$

$$+(b-a) = \lambda \cdot -\frac{1}{(a+b)}\left[-a^2 + \frac{(W_0-W_B)^2}{X^2}\right]$$

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$$\lambda = \frac{-(b^2-a^2)}{\left[-a^2 + \frac{(W_0-W_B)^2}{X^2}\right]}$$

λ AS A FUNCTION OF X

$$-\frac{(b^2-a^2)}{\lambda} + a^2 = \frac{(W_0-W_B)^2}{X^2}$$

$$X^2 = \frac{(W_0-W_B)^2}{a^2 - \frac{(b^2-a^2)}{\lambda}}$$

$$\boxed{X = \frac{W_0-W_B}{\pm\sqrt{a^2 - \frac{(b^2-a^2)}{\lambda}}}}$$

X AS FUNCTION OF λ

IF $W_0 > W_B$, WE TAKE $+$ BECAUSE WE ARE INTERESTED IN POSITIVE X.

NOTE THAT $a^2 > \frac{b^2-a^2}{\lambda}$ DUE TO EXISTENCE CONDITION FOR X^*

$$X^* = r - \frac{W_0-W_B}{X^*}$$

$$\boxed{X^* = r \mp \sqrt{a^2 - \frac{(b^2-a^2)}{\lambda}}}$$

X^* AS FUNCTION OF λ

$$(r-X^*)^2 = a^2 - \frac{(b^2-a^2)}{\lambda}$$

$$\boxed{\frac{-(b^2-a^2)}{(r-X^*)^2-a^2} = \lambda}$$

λ AS FUNCTION OF X^*

WE NOW INTRODUCE γ

$$F(x) = \begin{cases} 0 & x < r-a \\ \int_{r-a}^{x} f(s)\,ds = \left[\frac{s}{a+b}\right]_{r-a}^{x} = \frac{x-r+a}{a+b} & r-a \le x \le r+b \\ 1 & x > r+b \end{cases}$$

WHEN $\gamma \in (0,1)$ $\quad F(x^*) = \gamma \Rightarrow \boxed{\frac{x^*-r+a}{a+b} = \gamma}$

γ AS FUNCTION OF x^*

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$$\boxed{x^* = \gamma(a+b) + r - a}$$

x^* AS FUNCTION OF γ

$$\lambda = \frac{-(b^2-a^2)}{(a-\gamma(a+b))^2 - a^2} = \frac{-(b^2-a^2)}{\gamma^2(a+b)^2 - 2a\gamma(a+b)} = \frac{-(b-a)}{\gamma^2(a+b) - 2a\gamma}$$

λ AS FUNCTION OF γ

<u>THEREFORE</u>: CHOOSING γ AS THE PROBABILITY TO NOT REACH THE BENCHMARK, WE FIND λ_γ AND MAY THUS DEFINE THE UTILITY

$U(x) = x - \lambda_\gamma \max(W_B - x, 0)$

AND WE FIND, <u>IF</u> $b^2 - a^2 < a^2 \lambda_\gamma$, THE OPTIMAL ALLOCATION X^* AND THE MINIMUM RATE OF RETURN FOR THE RISKY ASSET X^* TO REACH THE BENCHMARK.

NOTE THAT $\gamma \to 0$ IMPLIES $\lambda_\gamma \to +\infty$ AND THIS MEANS THAT REDUCING THE RISK TO ZERO IMPLIES GIVING MORE AND MORE WEIGHT TO THE $-\max(W_B - X, 0)$ PENALTY PART OF U FUNCTION

AND WHEN $\gamma \to 1$, THIS MEANS THAT THE INVESTOR IS READY TO ACCEPT A 100% PROBABILITY OF NOT REACHING THE BENCHMARK, $\lambda_\gamma \to 0$ AND $U(x) = X$