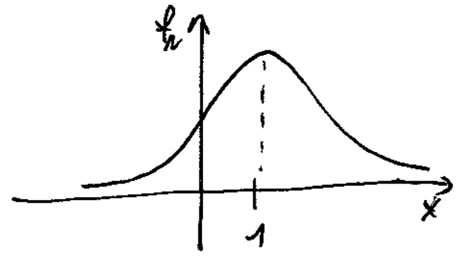
11.1 STOCHASTIC DOMINANCE

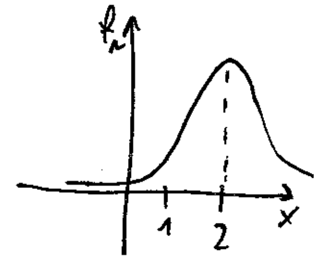
CONSIDER $L_1 = (0, 10\%; 1, 90\%)$ AND
$L_2 = (0, 10\%; 2, 90\%)$.

EVERYBODY, PROVIDED U IS INCREASING AND <u>NO</u> OTHER CONDITION, WILL PREFER L_2 OVER L_1

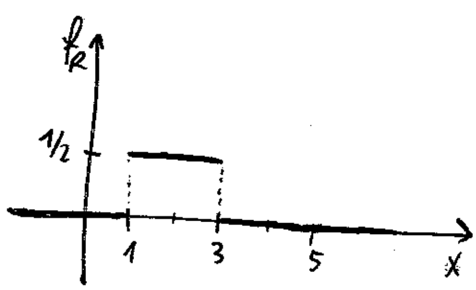
THE SAME FOR $L_1 = (0, 10\%; 1, 90\%)$ $L_2 = (0, 5\%; 1, 95\%)$

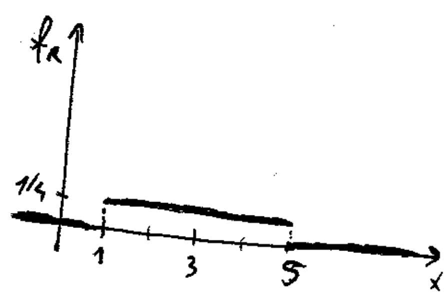
OR FOR CONTINOUS LOTTERIES $L_1 = N(1; \sigma^2)$ $L_2 = N(2; \sigma^2)$





OR FOR SLIGHTLY MORE COMPLEX SITUATIONS
$L_1 = R(1, 3)$ $L_2 = R(1, 5)$





THEREFORE IN MANY SITUATIONS WE MAY DECIDE WHICH LOTTERY IS BETTER. HOWEVER, IN OTHER SITUATIONS THIS DECISION MAY NOT BE TAKEN, FOR EXAMPLE

$L_1 = (0, 10\%; 1, 90\%)$ $L_2 = (-1, 10\%; 2, 90\%)$

WITHOUT KNOWING U, WE MAY NOT CHOOSE BETWEEN L_1 AND L_2

11.2

FIRST ORDER STOCHASTIC DOMINANCE FSD

L^1 FSD L^2 WHEN $F_1(X) < F_2(X)$ $\forall x$ SUCH THAT $F_1(X) \neq 0$, $F_1(X) \neq 1$, $F_2(X) \neq 0$, $F_2(X) \neq 1$

WHERE F IS THE CUMULATIVE DISTRIBUTION OF f...

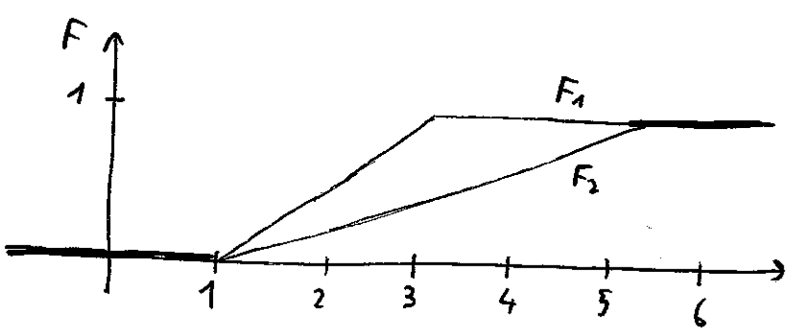
SO, EXCLUDING THOSE POINTS FOR WHICH THE CUMULATIVE DISTRIBUTION IS AT ITS MAXIMUM 1 OR MINIMUM 0, THE GRAPH OF 1 MUST STAY BELOW THE GRAPH OF 2

<u>EXAMPLE</u> : $L_1 = R(1,3)$ $L_2 = R(1,5)$

$$f_1 = \begin{cases} 0 & x<1 \\ 1/2 & 1 \leq x \leq 3 \\ 0 & x>3 \end{cases} \qquad f_2 = \begin{cases} 0 & x<1 \\ 1/4 & 1 \leq x \leq 5 \\ 0 & x>5 \end{cases}$$

$$F_1 = \begin{cases} 0 & x<1 \\ \int_1^x \frac{1}{2}\, ds & 1 \leq x \leq 3 \\ & x>3 \end{cases} \qquad F_2 = \begin{cases} 0 & x<1 \\ \int_1^x 1/4 \, ds & 1 \leq x \leq 5 \\ 1 & x>5 \end{cases}$$

$$F_1 = \begin{cases} 0 & x<1 \\ \frac{x-1}{2} & 1 \leq x \leq 3 \\ 1 & x>3 \end{cases} \qquad F_2 = \begin{cases} 0 & x<1 \\ \frac{x-1}{4} & 1 \leq x \leq 5 \\ 1 & x>5 \end{cases}$$



$F_1(x) = \frac{x-1}{2}$ WHERE IT IS NOT 0 NOR 1

$F_2(x) = \frac{x-1}{4}$ WHERE IT IS NOT 0 NOR 1 $\Rightarrow F_1(x) > F_2(x)$

$\Downarrow$

L_2 FIRST ORDER STOCHASTIC DOMINATES L_1

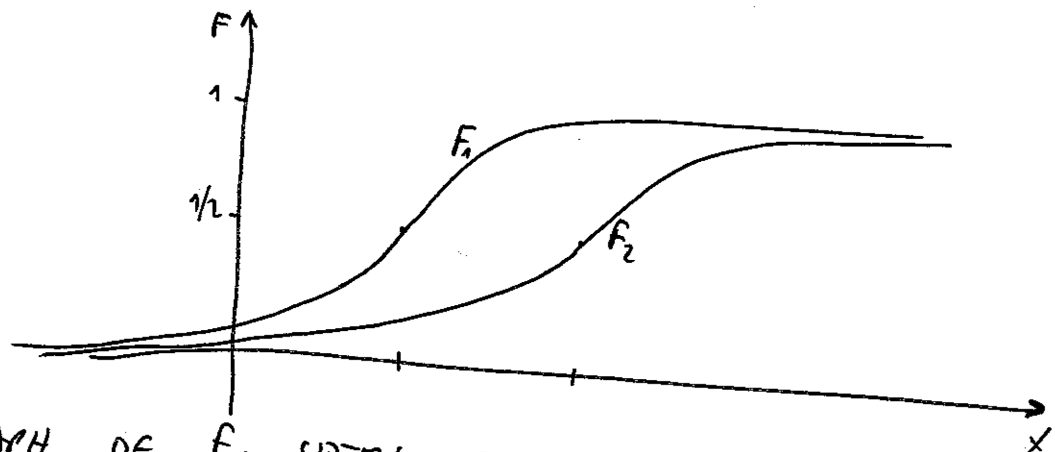
11.3

EXAMPLE $L_1 \sim N(1, \theta^2)$

$L_2 \sim N(2, \theta^2)$

IS THERE F.S.D. ?

WE DO NOT KNOW THE EXACT FORMULA FOR F_1 AND F_2, BUT WE KNOW THEIR SHAPES AND THE FACT THAT $F_1(1) = 1/2$ AND $F_2(2) = 1/2$ SINCE 1 AND 2 ARE THE MEDIANS



GRAPH OF F_2 SEEMS TO BE ALWAYS BELOW GRAPH OF F_1
LET'S CHECK ANALITICALLY :

$$F_1(x) = \phi\left(\frac{x-1}{\theta^2}\right) \qquad F_2(x) = \phi\left(\frac{x-2}{\theta^2}\right)$$

IS $\phi\left(\frac{x-2}{\theta^2}\right) \overset{?}{<} \phi\left(\frac{x-1}{\theta^2}\right)$ $\forall x$? SINCE ϕ IS INCREASING, THIS IS EQUIVALENT TO $\frac{x-2}{\theta^2} \overset{?}{<} \frac{x-1}{\theta^2}$ $\forall x$ $\Leftrightarrow$ $x-2 \overset{?}{<} x-1$ $\forall x$

$\Leftrightarrow$ $-2 \overset{?}{<} -1$ <u>OK</u>

$$\Downarrow$$

$F_1(x) > F_2(x)$ $\forall x$ $\quad L_2$ FSD L_1

IN GENERAL IT IS TRUE FOR EVERY $N_1(\mu_1, \theta^2)$ $N_2(\mu_2, \theta^2)$
$\mu_1 > \mu_2 \Rightarrow N_1$ FSD N_2

11.4

EXAMPLE: L_1 = EXPONENTIAL(λ) L_2 = EXPONENTIAL(μ) WITH $\mu > \lambda$

$$f_1 = \lambda e^{-\lambda x} \qquad f_2 = e^{-\mu x}\mu$$

$$F_1 = \int_0^x \lambda e^{-\lambda s} ds = \left[-e^{-\lambda s}\right]_0^x = -e^{-\lambda x} + 1 = 1 - e^{-\lambda x} \qquad x \in [0, +\infty)$$

$$F_2 = 1 - e^{-\mu x} \qquad x \in [0, +\infty)$$

$\mu > \lambda \Rightarrow -\mu < -\lambda$ SINCE $x > 0 \Rightarrow -\mu x < -\lambda x \Rightarrow e^{-\mu x} < e^{-\lambda x} \Rightarrow$

$\Rightarrow -e^{-\mu x} > -e^{-\lambda x} \Rightarrow 1 - e^{-\mu x} > 1 - e^{-\lambda x}$ (FOR $x \neq 0$)

$\Downarrow$

L_1 FIRST ORDER STOCHASTIC DOMINATES L_2

11.5

L_1 SECOND ORDER STOCHASTIC DOMINATES L_2

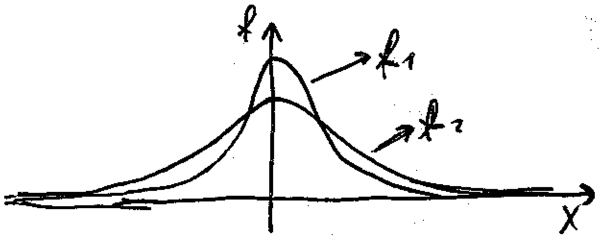
SSD

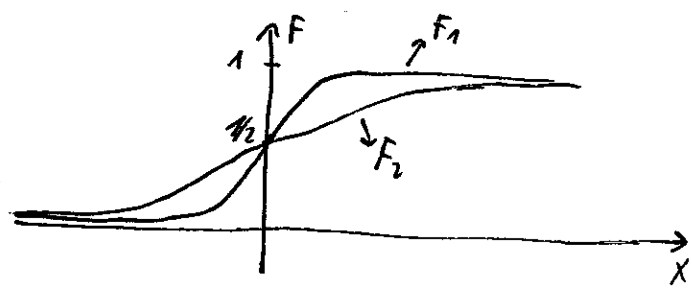
WHEN $\Delta(x) = \int_{-\infty}^{x} \left[F_1(s) - F_2(s)\right] ds < 0 \quad \forall x$ SUCH THAT $F_1(x) \neq 0$ $F_1(x) \neq 1$, $F_2(x) \neq 0$ $F_2(x) \neq 1$

THEOREM: $FSD \Rightarrow SSD$

OBVIOUS, SINCE $F_1 < F_2 \Rightarrow F_1 - F_2 < 0 \Rightarrow \int_{-\infty}^{x} F_1 - F_2 < 0$

DIFFICULT EXAMPLE: $L_1 = N(0, \theta_1^2)$ $L_2 = N(0, \theta_2^2)$ $\theta_1^2 < \theta_2^2$





FROM THE GRAPH IT IS OBVIOUS THAT THERE IS NO FIRST ORDER STOCHASTIC DOMINANCE!

TO PROVE IT USING THE INEQUALITY, WE INTRODUCE

$\phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds$. IF WE CALCULATE $\phi\left(\frac{x}{\sigma}\right) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{s^2}{2\sigma^2}} ds$

LET'S CHECK $F_1(x)$ AND $F_2(x)$

THEREFORE WE CONFRONT $F_1 = \phi\left(\frac{x}{\theta_1}\right)$ AND $F_2 = \phi\left(\frac{x}{\theta_2}\right)$

FOR $x > 0$ $\quad \frac{x}{\theta_1} > \frac{x}{\theta_2} \Rightarrow \phi\left(\frac{x}{\theta_1}\right) > \phi\left(\frac{x}{\theta_2}\right)$ SINCE ϕ IS INCREASING

FOR $x < 0$ $\quad \frac{x}{\theta_1} < \frac{x}{\theta_2} \Rightarrow \phi\left(\frac{x}{\theta_1}\right) < \phi\left(\frac{x}{\theta_2}\right)$

THEREFORE FOR DIFFERENT x IT HAS A DIFFERENT INEQUALITY!

11.6 LET'S CHECK FOR SECOND ORDER S.O.

$$\Delta(x) = \int_{-\infty}^{x} F_1(s) - F_2(s)\, ds \qquad \forall x \in (-\infty, +\infty)$$

TAKE $x \geq 0$ AND WE HAVE

$$\Delta(x) = \int_{-\infty}^{-x} F_1(s) - F_2(s)\, ds + \int_{-x}^{x} F_1(s) - F_2(s)\, ds$$

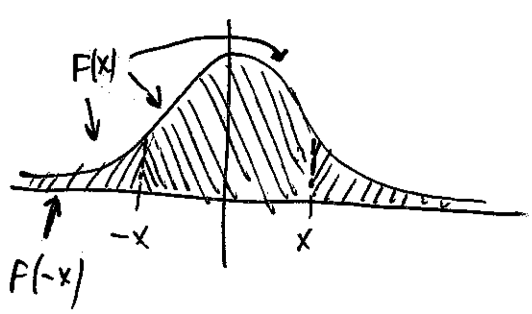
WE DEFINE $I(x) = \int_{-x}^{x} F_1(s) - F_2(s)\, ds$

AND $\Delta(x) = \Delta(-x) + I(x)$

NOW $I(x) = \int_{-x}^{x} F_1(s) - F_2(s)\, ds = \int_{-x}^{0} F_1(s) - F_2(s)\, ds + \int_{0}^{x} F_1(s) - F_2(s)\, ds$

AND DEFINE $J(x) = \int_{0}^{x} F_1(s) - F_2(s)\, ds$

NOW WE USE A PROPERTY OF CUMULATIVE DISTRIBUTION FOR SYMMETRIC DENSITIES : $F(x) = 1 - F(-x)$



$$J(x) = \int_0^x F_1(s) - F_2(s)\, ds = \int_0^x (1 - F_1(-s)) - (1 - F_2(-s))\, ds =$$

$$= -\int_0^x F_1(-s) - F_2(-s)\, ds \underset{\substack{\text{CHANGE} \\ \text{VARIABLE} \\ z = -s}}{=} -\int_0^{-x} F_1(z) - F_2(z) \cdot -1 \cdot dz = -\int_{-x}^{0} F_1(z) - F_2(z)\, dz$$

$$\int_{-x}^{0} F_1(z) - F_2(z)\, dz = -J(x)$$

THEREFORE $I(x) = -J(x) + J(x) = 0 \implies \Delta(x) = \Delta(-x) \quad \forall x \geq 0$

11.7

THEREFORE WE JUST NEED TO CHECK THE SIGN OF $\Delta(x)$ FOR NEGATIVE X, AND POSITIVE X WILL HAVE A Δ WITH THE SAME SIGN

$X \leq 0$ LET'S USE FUNCTION $\phi(x) = \int_{-\infty}^{x} f_{N(0,1)}(s)\, ds$

WHICH HAS THE PROPRETY SUCH THAT $\phi\left(\frac{x}{\sigma}\right) = \int_{-\infty}^{x} f_{N(0,\sigma^2)}(s)\, ds$

$\Delta(x) = \int_{-\infty}^{x} \phi\left(\frac{s}{\sigma_1}\right) - \phi\left(\frac{s}{\sigma_2}\right) ds$ WITH $x \leq 0 \Rightarrow s \leq 0$

$\sigma_1^2 < \sigma_2^2 \Rightarrow \sigma_1 < \sigma_2 \Rightarrow \frac{1}{\sigma_1} > \frac{1}{\sigma_2} \Rightarrow \frac{s}{\sigma_1} < \frac{s}{\sigma_2} \Rightarrow \phi\left(\frac{s}{\sigma_1}\right) < \phi\left(\frac{s}{\sigma_2}\right) \Rightarrow$

$\Rightarrow$ THE FUNCTION $\phi\left(\frac{s}{\sigma_1}\right) - \phi\left(\frac{s}{\sigma_2}\right) \underset{\forall s \leq 0}{< 0} \Rightarrow \Delta(x) < 0 \ \forall x \leq 0$

SO $\Delta(x) < 0$ AND L_1 SSD L_2

<u>TECHNICAL NOTE:</u>

TO BE SURE OF THIS, WE MUST CHECK THAT THE INTEGRAL EXISTS:

$\Delta(x) = \int_{-\infty}^{x} F_1(s) - F_2(s)\, ds$ WHEN $s \to -\infty$ $F_1(s) \to 0$, $F_2(s) \to 0$ SINCE THEY ARE CUMULATIVE FUNCTIONS OF THE NORMAL, AND THEREFORE

$\lim_{k \to -\infty} \int_{-k}^{x} F_1(s) - F_2(s)\, ds$ EXISTS!

(THERE WOULD HAVE BEEN PROBLEMS IF $F_1 \to +\infty$ $F_2 \to +\infty$ BECAUSE WE WOULD HAVE $+\infty - \infty$ UNDER THE INTEGRAL!)

$\square$

11-8 $L_1 \sim N(0,2)$ $\quad L_2 \sim N(0,5)$

WHICH IS MORE RISKY?

LET'S CHECK FSD

$$F_1 = \phi\left(\frac{x-0}{\sqrt{2}}\right) \qquad F_2 = \phi\left(\frac{x-0}{\sqrt{5}}\right)$$

CHECK FSD IN $(-\infty, +\infty)$

$F_1(x) < F_2(x)$ IS NOT TRUE FOR POSITIVE x

$F_1(x) > F_2(x)$ IS NOT TRUE FOR NEGATIVE x

LET'S CHECK SSD

$$\Delta(x) = \int_{-\infty}^{x} \phi\left(\frac{s}{\sqrt{2}}\right) - \phi\left(\frac{s}{\sqrt{5}}\right) ds \qquad \text{FOR } x \in (-\infty, +\infty)$$

FOR POSITIVE X WE HAVE

$$\Delta(x) = \int_{-\infty}^{-x} \phi\left(\frac{s}{\sqrt{2}}\right) - \phi\left(\frac{s}{\sqrt{5}}\right) ds + \int_{-x}^{0} \phi\left(\frac{s}{\sqrt{2}}\right) - \phi\left(\frac{s}{\sqrt{5}}\right) ds + \int_{0}^{x} \phi\left(\frac{s}{\sqrt{2}}\right) - \phi\left(\frac{s}{\sqrt{5}}\right) ds =$$

$t = -s$ $\qquad \phi(x) = 1 - \phi(-x)$

$$= \Delta(-x) + \int_{x}^{0} \phi\left(-\frac{t}{\sqrt{2}}\right) - \phi\left(-\frac{t}{\sqrt{5}}\right)(-dt) + \int_{0}^{x} \cancel{1} - \phi\left(-\frac{s}{\sqrt{2}}\right) - \left(\cancel{1} - \phi\left(-\frac{s}{\sqrt{5}}\right)\right) ds =$$

$$= \Delta(-x) + \int_{0}^{x} \phi\left(-\frac{t}{\sqrt{2}}\right) - \phi\left(-\frac{t}{\sqrt{5}}\right) dt - \int_{0}^{x} \phi\left(-\frac{s}{\sqrt{2}}\right) - \phi\left(-\frac{s}{\sqrt{5}}\right) ds = \Delta(-x)$$

THEREFORE WE JUST NEED TO CHECK THE SIGN OF $\Delta(x)$ FOR $X \leq 0$

BUT FOR $X \leq 0$ $\quad \phi\left(\frac{x}{\sqrt{2}}\right) \leq \phi\left(\frac{x}{\sqrt{5}}\right)$ $\quad$ SINCE $\frac{x}{\sqrt{2}} \leq \frac{x}{\sqrt{5}}$ FOR $x \leq 0$

THEREFORE $\phi\left(\frac{s}{\sqrt{2}}\right) - \phi\left(\frac{s}{\sqrt{5}}\right) \leq 0 \quad \forall s \leq 0$

AND $\phi\left(\frac{s}{\sqrt{2}}\right) - \phi\left(\frac{s}{\sqrt{5}}\right) < 0 \quad \forall s < 0$ $\implies \displaystyle\int_{-\infty}^{x} \phi\left(\frac{s}{\sqrt{2}}\right) - \phi\left(\frac{s}{\sqrt{5}}\right) ds < 0 \quad \forall x \leq 0$

L_1 SSD $L_2 \iff L_2$ IS MORE RISKY $\square$

11.8 WHAT IS THE MEANING OF FSD AND SSD

CONSIDER $f_1(x), f_2(x)$ DENSITIES DEFINED ON (a,b)

$$U(L^1) - U(L^2) = \int_a^b u(s) f_1(s)\, ds - \int_a^b u(s) f_2(s)\, ds =$$

IF U' EXISTS, INTEGRATING BY PARTS

$$= \left[U(s) F_1(s)\right]_a^b - \int_a^b U'(s) F_1(s) ds - \left[U(s) F_2(s)\right]_a^b + \int_a^b U'(s) F_2(s) ds =$$

$$= U(b) F_1(b) - U(a) F_1(a) - U(b) F_2(b) + U(a) F_2(a) - \int_a^b U'(s)(F_1(s) - F_2(s))\, ds =$$

SINCE $f_i(x)$ ARE DEFINED ON (a,b), OUTSIDE THEY ARE ZERO AND

$$F_i(x) = \int_{-\infty}^{x} f_i(s) ds \qquad F_i(a) = \int_{-\infty}^{a} f_i(s) ds = \int_{-\infty}^{a} 0\, ds = 0 \qquad F_i(b) = \int_{-\infty}^{b} f_i(s) ds = \int_{-\infty}^{a} 0\, ds + \int_a^b f ds = 1$$

$$= \cancel{U(b)} \cdot 1 - 0 - \cancel{U(b)} \cdot 1 + 0 - \int_a^b U'(F_1 - F_2) ds$$

L_1 FSD $L_2 \Leftrightarrow F_1 < F_2 \;\forall x \in (a,b) \Leftrightarrow F_1 - F_2 < 0 \;\forall x \in (a,b)$

AND SINCE $U' > 0$, SINCE U IS UTILITY, $U(L_1) - U(L_2) > 0$

AND $U(L_1) > U(L_2)$ AND AGENT PREFERS L^1 OVER L^2

IF U'' EXISTS, TAKING

$$U(L^1) - U(L^2) = -\int_a^b U'(s)(F_1(s) - F_2(s))\, ds = \quad \text{AND INTEGRATING BY PARTS}$$

$$= \left[-U'(s) \cdot \left(\int_a^s F_1(t) - F_2(t)\, dt\right)\right]_a^b + \int_a^b U''(s) \cdot \left(\int_a^s F_1(t) - F_2(t)\, dt\right) ds =$$

$$= -U'(b) \cdot \left(\int_a^b F_1(t)\, dt - \int_a^b F_2(t)\, dt\right) + U'(a) \cdot \cancel{\left(\int_a^a F_1 - F_2\, dt\right)} + \int_a^b U''(s) \cdot \left(\int_a^s F_1(t) - F_2(t)\, dt\right) ds$$

$\boxed{11.9}$

$$= \underset{\overset{\wedge}{0}}{-U'(b)} \cdot \int_0^b F_1 - F_2 + 0 + \int_0^b U'' \cdot \left(\int_0^s F_1 - F_2 \right) ds$$

L_1 SSD $L_2 \iff \int_0^s F_1(t) - F_2(t)\, dt < 0 \;\; \forall s \in (0,b)$, ALSO FOR $\int_0^b F_1 - F_2 < 0$

AND IF AGENT IS <u>STRICTLY RISK-AVERSE</u> $\iff U'' < 0$

$$\Downarrow$$

$$U(L^1) - U(L^2) \geq 0$$

$$\Downarrow$$

AGENT PREFERS L^1 OVER L^2

11.10

CONSIDER TWO LOGNORMAL DISTRIBUTIONS

$LN(\mu, \theta_1^2)$ $\quad LN(\mu, \theta_2^2)$ $\quad \theta_1^2 > \theta_2^2$

- WHICH ONE is FSD?
- WHICH ONE is SSD?

$$F_{LN(\mu,\theta_1^2)} = \phi\left(\frac{\ln x - \mu}{\theta_1}\right) \qquad F_{LN(\mu,\theta_2^2)} = \phi\left(\frac{\ln x - \mu}{\theta_2}\right)$$

FOR $x > e^{\mu} \Rightarrow \ln x - \mu > 0 \Rightarrow \frac{\ln x - \mu}{\theta_1} < \frac{\ln x - \mu}{\theta_2}$ AND SINCE ϕ is INCREASING

$\Downarrow$

$F_{LN(\mu,\theta_1)} < F_{LN(\mu,\theta_2)}$

FOR $x < e^{\mu} \Rightarrow \ln x - \mu < 0 \Rightarrow \frac{\ln x - \mu}{\theta_1} > \frac{\ln x - \mu}{\theta_2} \Rightarrow F_{LN(\mu,\theta_1)} > F_{LN(\mu,\theta_2)}$

THEREFORE THERE is NO FSD

LET'S SEE SSD

$$\Delta(x) = \int_0^x \left[F_{LN(\mu,\theta_1^2)}(t) - F_{LN(\mu,\theta_2^2)}(t)\right] dt = \int_0^x \phi\left(\frac{\ln t - \mu}{\theta_1}\right) - \phi\left(\frac{\ln t - \mu}{\theta_2}\right) dt$$

NOTE THAT $x \geq 0$

$\Delta(0) = 0 \qquad \Delta'(x) = \phi\left(\frac{\ln x - \mu}{\theta_1}\right) - \phi\left(\frac{\ln x - \mu}{\theta_2}\right)$

$\Delta'(x) > 0$ WHEN $x < e^{\mu}$

$\Delta'(x) < 0$ WHEN $x > e^{\mu}$

$\Delta'(x) = 0$ WHEN $x = e^{\mu}$

SO IN $x = e^{\mu}$ $\Delta(x)$ HAS A RELATIVE MAXIMUM

11.11 LET'S SEE WHAT HAPPENS FOR $X \to +\infty$

CONSIDER $$\int_0^x \phi\left(\frac{\ln t-\mu}{\sigma}\right) dt \underset{\substack{\text{SUBSTITUTE}\\ z=\ln t-\mu}}{=} \int_{-\infty}^{\ln x-\mu} \phi\left(\frac{z}{\sigma}\right) e^{z+\mu}\, dz =$$

$$\underset{\text{BY PARTS}}{=} e^{\mu}\left[e^{z}\phi\left(\frac{z}{\sigma}\right)\right]_{-\infty}^{\ln x-\mu} - e^{\mu}\int_{-\infty}^{\ln x-\mu} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} e^{z}\, dz$$

$$\lim_{x\to+\infty} \Delta(x) = \lim_{x\to+\infty}\int_0^x \phi\left(\frac{\ln t-\mu}{\sigma_1}\right)-\phi\left(\frac{\ln t-\mu}{\sigma_2}\right) dt = \lim_{x\to+\infty} e^{\mu}\left\{\left[e^{z}\phi\left(\frac{z}{\sigma_1}\right)-e^{z}\phi\left(\frac{z}{\sigma_2}\right)\right]_{-\infty}^{\ln x-\mu}\right\}$$

$$+\lim_{x\to+\infty} -e^{\mu}\int_{-\infty}^{\ln x-\mu} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{z^2}{2\sigma_1^2}} e^{z} - \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{z^2}{2\sigma_2^2}} e^{z}\, dz =$$

$$= e^{\mu}\lim_{z\to+\infty} e^{z}\left(\phi\left(\frac{z}{\sigma_1}\right)-\phi\left(\frac{z}{\sigma_2}\right)\right) - e^{\mu}\lim_{z\to-\infty} e^{z}\left(\phi\left(\frac{z}{\sigma_1}\right)-\phi\left(\frac{z}{\sigma_2}\right)\right) + J =$$

$$= e^{\mu}\cdot 0 - e^{\mu}\cdot 0 + J = J$$

LET'S ANALYZE J $$\int_{-\infty}^{\ln x-\mu} e^{-\frac{z^2-2\sigma_1^2 z}{2\sigma_1^2}}\, dz = \int_{-\infty}^{\ln x-\mu} e^{-\frac{z^2-2\sigma_1^2 z+\sigma_1^4}{2\sigma_1^2}}\cdot e^{\frac{\sigma_1^2}{2}}\, dz$$

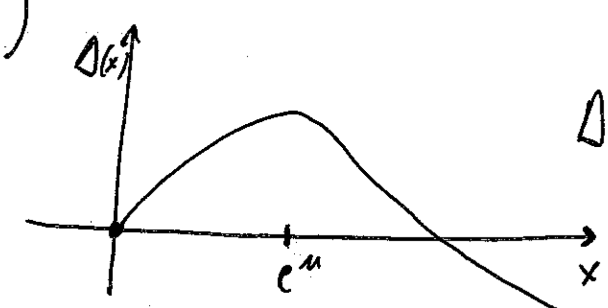
$$= \int_{-\infty}^{\ln x-\mu} e^{-\frac{(z-\sigma_1^2)^2}{2\sigma_1^2}}\cdot e^{\frac{\sigma_1^2}{2}}\, dz \underset{\substack{\text{SUBSTITUTION}\\ v=\frac{z-\sigma_1^2}{\sqrt{2\sigma_1^2}}}}{=} e^{\frac{\sigma_1^2}{2}}\sqrt{2\sigma_1^2}\int_{-\infty}^{\frac{\ln x-\mu-\sigma_1^2}{\sqrt{2\sigma_1^2}}} e^{-v^2}\, dv \xrightarrow[x\to+\infty]{}$$

$$\xrightarrow[x\to+\infty]{} e^{\frac{\sigma_1^2}{2}}\sqrt{2\sigma_1^2}\cdot\sqrt{\pi}$$

$$J = -\frac{e^{\mu}}{\cancel{\sqrt{2\pi\sigma_1^2}}} e^{\frac{\sigma_1^2}{2}} \cancel{\sqrt{2\sigma_1^2}}\cancel{\sqrt{\pi}} + \frac{e^{\mu}}{\cancel{\sqrt{2\pi\sigma_2^2}}} e^{\frac{\sigma_2^2}{2}}\cancel{\sqrt{2\sigma_2^2}}\cancel{\sqrt{\pi}} =$$

$$= e^{\mu}\left\{-e^{\frac{\sigma_1^2}{2}} + e^{\frac{\sigma_2^2}{2}}\right\} < 0$$

THEREFORE



$\Delta(x)$ BECOMES NEGATIVE FOR LARGE X

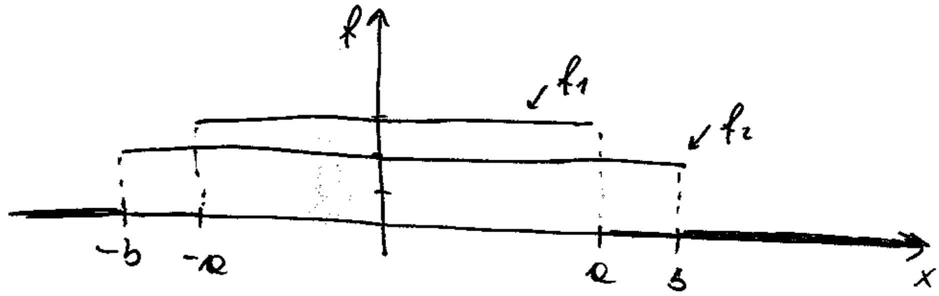
THERE IS NO SSD $\square$

11.12) EXERCISE

$$R_1(-a,a) \qquad R_2(-b,b) \qquad 0<a<b$$

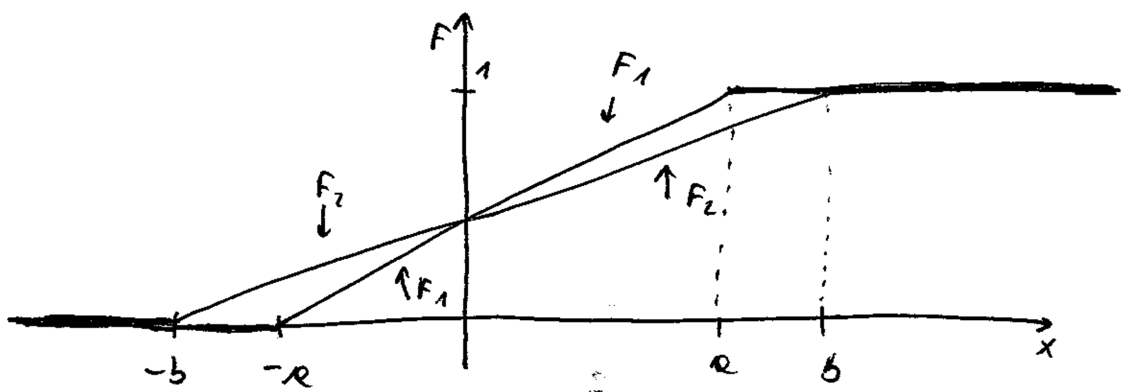
- WHICH is FSD ?
- WHICH is SSD

$$f_1=\begin{cases}0 & x<-a\\ \frac{1}{2a} & -a\le x\le a\\ 0 & x>a\end{cases} \qquad f_2=\begin{cases}0 & x<-b\\ \frac{1}{2b} & -b\le x\le b\\ 0 & x>b\end{cases}$$



$$F_1=\begin{cases}0 & x<-a\\ \int_{-a}^{x}\frac{1}{2a}\,ds & -a\le x\le a\\ 1 & x>a\end{cases} \qquad F_2=\begin{cases}0 & x<-b\\ \int_{-b}^{x}\frac{1}{2b}\,ds & -b\le x\le b\\ 1 & x>b\end{cases}$$

$$\int_{-a}^{x}\frac{1}{2a}\,ds=\left[\frac{1}{2a}\cdot s\right]_{-a}^{x}=\frac{x}{2a}+\frac{1}{2} \qquad \int_{-b}^{x}\frac{1}{2b}\,ds=\frac{x}{2b}+\frac{1}{2}$$



THERE IS NO FSD.

IN FACT FOR $x\in(-a,0)$ $\quad F_1=\frac{x}{2a}+\frac{1}{2} \underset{\substack{\text{SINCE}\\ x<0}}{<} \frac{x}{2b}+\frac{1}{2}=F_2$

FOR $x\in(0,a)$ $\quad F_1=\frac{x}{2a}+\frac{1}{2}>\frac{x}{2b}+\frac{1}{2}=F_2$

11.13

LET'S CHECK S.S.D

$$\Delta(x) = \int_{-b}^{x} F_1(t) - F_2(t)\, dt$$

FOR $x \le -a$, $x \ge a$ WE ARE NOT INTERESTED

FOR $x \in (-a, a)$
$$\Delta(x) = \int_{-b}^{-a} 0 - \frac{s}{2b} - \frac{1}{2}\, ds + \int_{-a}^{x} \frac{s}{2a} + \frac{1}{2} - \frac{s}{2b} - \frac{1}{2}\, ds =$$

$$= \left[-\frac{s^2}{4b} - \frac{s}{2} \right]_{-b}^{-a} + \left[\frac{s^2}{4a} - \frac{s^2}{4b} \right]_{-a}^{x} = -\frac{a^2 - b^2}{4b} + \frac{a}{2} - \frac{b}{2} + \frac{x^2}{4a} - \frac{x^2}{4b} - \frac{a}{4} + \frac{a^2}{4b} =$$

$$= -\frac{b}{4} + \frac{a}{4} + \frac{x^2}{4a} - \frac{x^2}{4b} = x^2\left(\frac{b-a}{4ab}\right) + \frac{a-b}{4}.$$

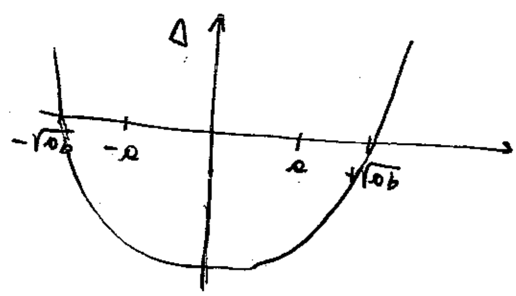
SECOND DEGREE POLYNOMIAL WITH SOLUTIONS $x_{1,2} = \pm\sqrt{\frac{b-a}{4} \cdot \frac{4ab}{b-a}}$

$x_{1,2} = \pm\sqrt{ab}$

$b > a$

$\Downarrow$

$|x_{1,2}| > a$



THEREFORE $\Delta(x) < 0$ WITH $x \in (-a, a)$

$\Downarrow$

R_1 S.S.D R_2

11.14

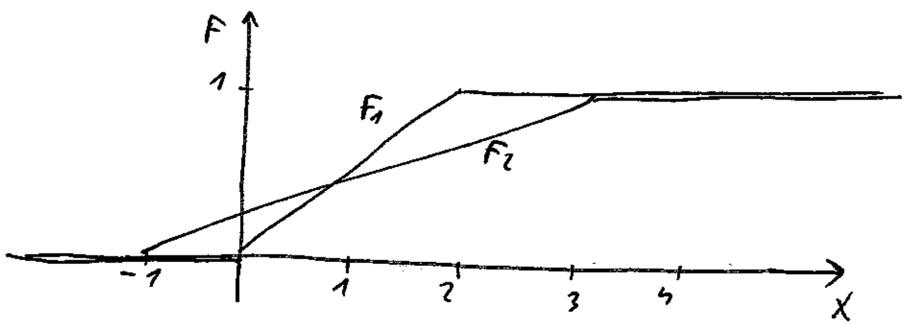
EXERCISE

$L_1 \sim R(0,2) \qquad L_2 \sim R(-1,3)$

WHICH ONE IS FSD? WHICH ONE IS SSD?

$$f_1 = \begin{cases} 0 & x<0 \\ 1/2 & 0 \le x \le 2 \\ 0 & x>2 \end{cases} \qquad f_2 = \begin{cases} 0 & x<-1 \\ 1/4 & -1 \le x \le 3 \\ 0 & x>3 \end{cases}$$

$$F_1 = \begin{cases} 0 & x<0 \\ \frac{x}{2} & 0 \le x \le 2 \\ 1 & x>2 \end{cases} \qquad F_2 = \begin{cases} 0 & x<-1 \\ \frac{x+1}{4} & -1 \le x \le 3 \\ 1 & x>3 \end{cases}$$



NO FSD.

$$\Delta(x) = \int_{-\infty}^{x} F_1(s) - F_2(s)\, ds =$$

WE MUST LOOK AT ITS SIGN IN THE AREA $x \in (0,2)$ ONLY

$$= \int_{-\infty}^{-1} 0 - 0\, ds + \int_{-1}^{0} 0 - \frac{s+1}{4}\, ds + \int_{0}^{x} \frac{s}{2} - \frac{s+1}{4}\, ds =$$

$$= 0 + \left[-\frac{(s+1)^2}{8} \right]_{-1}^{0} + \int_0^x \frac{2s - s - 1}{4}\, ds = -\frac{1}{8} + 0 + \int_0^x \frac{s-1}{4}\, ds =$$

$$= -\frac{1}{8} + \left[\frac{(s-1)^2}{8} \right]_0^x = -\frac{1}{8} + \frac{(x-1)^2}{8} - \frac{1}{8} = -\frac{1}{4} + \frac{x^2}{8} + \frac{1}{8} - \frac{x}{4} = \frac{x^2}{8} - \frac{x}{4} - \frac{1}{8}$$

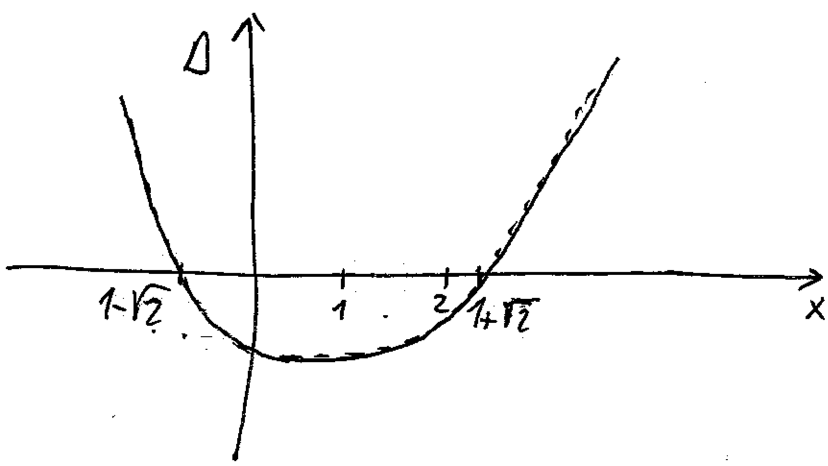
11.15

$$\Delta(x) = \frac{x^2}{8} - \frac{x}{4} - \frac{1}{8}$$

SECOND-DEGREE POLYNOMIAL

x^2 COEFFICIENT POSITIVE $\Rightarrow$ ∪

$$x_{1,2} = \frac{1/4 \pm \sqrt{\frac{1}{16} + 4 \cdot \frac{1}{8} \cdot \frac{1}{8}}}{\frac{1}{4}} = \frac{1/4 \pm \frac{\sqrt{2}}{4}}{\frac{1}{4}} = 1 \pm \sqrt{2}$$



THE AREA TO CHECK IS BETWEEN 0 AND 2

AND SINCE $1+\sqrt{2}$ IS APPROX. 2,41

WE HAVE THAT BETWEEN 0 AND 2 $\Delta(x) < 0$

$\Downarrow$

F_1 SSD F_2

11.16

THEOREM:

TAKE L^1 WITH $E(L^1)=\mu$ $VAR(L^1)=\theta_1^2$ AND $f_1(x)$

L^2 WITH $E(L^2)=\mu$ $VAR(L^2)=\theta_2^2$ AND $f_2(x)$

WITH f_1 AND f_2 DEFINED ON (a,b)

$$\theta_1^2-\theta_2^2=\int_a^b (x-\mu)^2 f_1(x)\,dx-\int_a^b (x-\mu)^2 f_2(x)\,dx=\int_a^b (x-\mu)^2\left(f_1(x)-f_2(x)\right)dx=$$

$$\underset{\text{INTEGRATION BY PARTS}}{=}\left[(x-\mu)^2\left(F_1(x)-F_2(x)\right)\right]_a^b-\int_a^b 2(x-\mu)\left(F_1(x)-F_2(x)\right)dx=$$

$$=(b-\mu)^2(1-1)-(a-\mu)^2(0-0)-2\int_a^b (x-\mu)\left(F_1(x)-F_2(x)\right)dx=$$

$$\underset{\text{INTEGRATION BY PARTS}}{=}\left[-2(x-\mu)\int_a^x F_1(t)-F_2(t)\,dt\right]_a^b+2\int_a^b 1\cdot\left(\int_a^x F_1(t)-F_2(t)\,dt\right)dx=$$

$$=-2(b-\mu)\int_a^b F_1-F_2+\cancel{2(a-\mu)\int_a^a F_1-F_2}_{=0}+2\int\left(\int_a^x F_1(t)-F_2(t)\,dt\right)dx$$

LET'S CALCULATE $\int_a^b F(x)\,dx$

$$\mu=\int_a^b x f_i(x)\underset{\text{INTEGRAT. BY PARTS}}{=}\left[x F_i(x)\right]_a^b-\int_a^b F_i(x)\,dx=bF_i(b)-aF_i(a)-\int_a^b F_i(x)\,dx=$$

$$=b\cdot 1-a\cdot 0-\int_a^b F_i(x)\,dx \Rightarrow \int_a^b F_i(x)\,dx=b-\mu \quad \forall i=1,2$$

$$\Downarrow$$

$$\int_a^b F_1-F_2=(b-\mu)-(b-\mu)=0$$

$$\theta_1^2-\theta_2^2=2\int_a^b\int_a^x F_1-F_2\,dt\,dx$$

IF L_1 SSD $L_2 \Rightarrow \int_a^x F_1-F_2\,dt<0 \Rightarrow \theta_1^2<\theta_2^2$

SSD FOR LOTTERIES WITH THE SAME EXPECTED VALUE IMPLIES SMALLER VARIANCE!

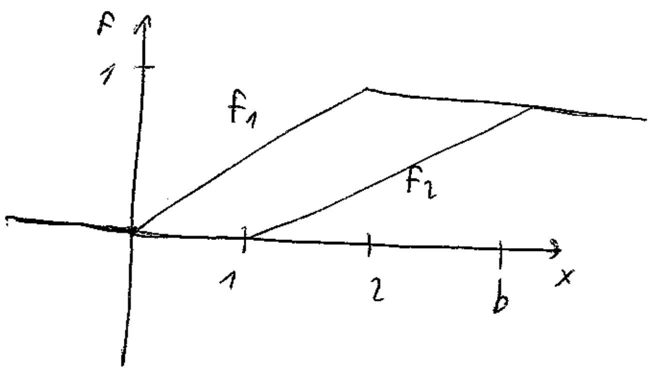
11.17

$L_1 \sim R(0,2)$ $\quad$ $L_2 \sim R(1,b)$

IS THERE FSD FOR $b \geq 3$?

$$f_1 = \begin{cases} 0 & x<0 \\ 1/2 & 0 \leq x \leq 2 \\ 0 & x>2 \end{cases} \qquad f_2 = \begin{cases} 0 & x<1 \\ 1/(b-1) & 1 \leq x \leq b \\ 0 & x>b \end{cases}$$

$$F_1 = \begin{cases} 0 & x<0 \\ \frac{x}{2} & 0 \leq x \leq 2 \\ 1 & x>2 \end{cases} \qquad F_2 = \begin{cases} 0 & x<1 \\ \frac{x-1}{b-1} & 1 \leq x \leq b \\ 1 & x>b \end{cases}$$



is $F_2 \overset{?}{<} F_1$?

BETWEEN 1 AND 2

$$\frac{x-1}{b-1} \overset{?}{<} \frac{x}{2} \qquad \forall x \in (1,2)$$

$$\frac{2x-1}{2(b-1)} \overset{?}{<} \frac{xb-x}{2(b-1)} \qquad \forall x \in (1,2)$$

SINCE $(b-1)>0$

$$3x \overset{?}{<} xb+1 \qquad (3-b)x \overset{?}{<} 1$$

IF $b \geq 3$ OK

F_2 FSD F_1

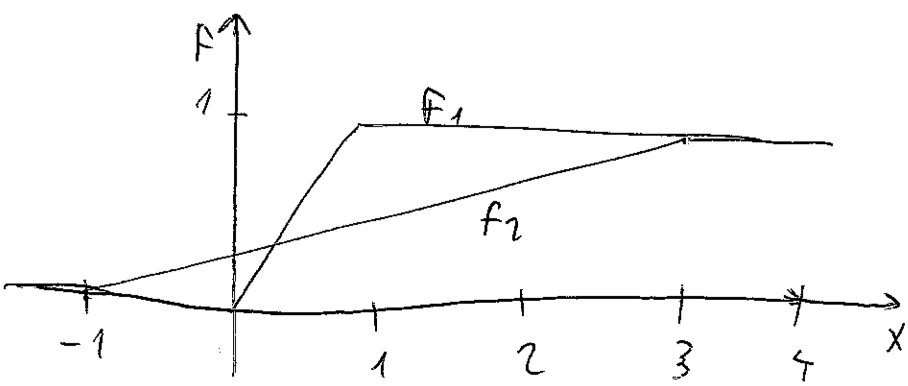
11.18

$L_1 \sim R(0,1)$ $\quad L_2 \sim R(-1;3)$

IS THERE FSD ? IS THERE SSD ?

$$f_1 = \begin{cases} 0 & x<0 \\ 1 & 0 \le x \le 1 \\ 0 & x>1 \end{cases} \qquad f_2 = \begin{cases} 0 & x<-1 \\ 1/4 & -1 \le x \le 3 \\ 0 & x>3 \end{cases}$$

$$F_1 = \begin{cases} 0 & x<0 \\ x & 0 \le x \le 1 \\ 1 & x>1 \end{cases} \qquad F_2 = \begin{cases} 0 & x<-1 \\ \frac{x+1}{4} & -1 \le x \le 3 \\ 1 & x>3 \end{cases}$$



NO FSD.

TO PROVE IT, FOR $x = 0.1$ $\quad F_1(0.1) < F_2(0.1)$

FOR $x = 0.9$ $\quad F_1(0.9) > F_2(0.9)$

$$\Delta(x) = \int_{-\infty}^{x} F_1(s) - F_2(s)\, ds = \int_{-\infty}^{-1} 0 - 0\, ds + \int_{-1}^{0} 0 - \frac{s+1}{4}\, ds + \int_{0}^{x} s - \frac{s+1}{4}\, ds =$$

$x \in (0,1)$

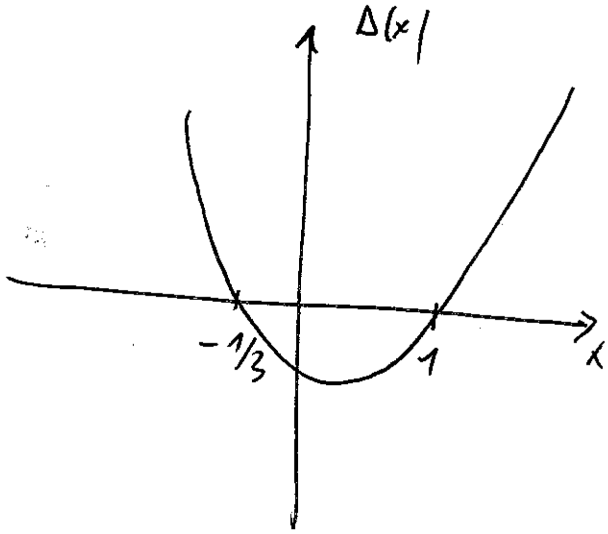
$$= -\left[\frac{(s+1)^2}{8}\right]_{-1}^{0} + \int_0^x \frac{3s-1}{4}\, ds = -\frac{1}{8} + \left[\frac{(3s-1)^2}{24}\right]_0^x = -\frac{1}{8} + \frac{(3x-1)^2}{24} - \frac{1}{24} =$$

11.19

$$= -\frac{4}{24} + \frac{9x^2}{24} + \frac{1}{24} - \frac{x}{4} = \frac{3}{8}x^2 - \frac{x}{4} - \frac{3}{24}$$

$$x_{1,2} = \frac{1/4 \pm \sqrt{\frac{1}{16} + \frac{9}{48}}}{\frac{3}{4}} = \frac{1/4 \pm \sqrt{\frac{12}{48}}}{3/4} = \frac{1/4 \pm 1/2}{3/4} =$$

$$= \frac{+1 \pm 2}{3} = \begin{cases} 1 \\ -\frac{1}{3} \end{cases}$$



$\Delta(x)$ SHOULD BE POS OR NEG $\forall x \in (0,1)$

IN FACT IT IS <u>NEG</u>

$\Downarrow$

F_1 SSD F_2

$\square$

11.20

$L_1 \sim N(-1, \sigma^2)$ $\quad L_2 \sim LN(0, \sigma^2)$

IS THERE FSD ?

$F_1 = \phi\left(\frac{x+1}{\sigma}\right)$ $\quad F_2 = \phi\left(\frac{\ln x}{\sigma}\right)$ $\quad \phi$ INCREASING

WE HAVE TO COMPARE THEREFORE

$\frac{x+1}{\sigma}$ AND $\frac{\ln x}{\sigma}$

WHICH MEANS COMPARING

$x+1$ AND $\ln x$ $\quad \forall x \in (0, +\infty)$

FOR VALUES CLOSE TO 0 $\quad x+1>0$ WHILE $\ln x < 0$

THEREFORE PROBABLY

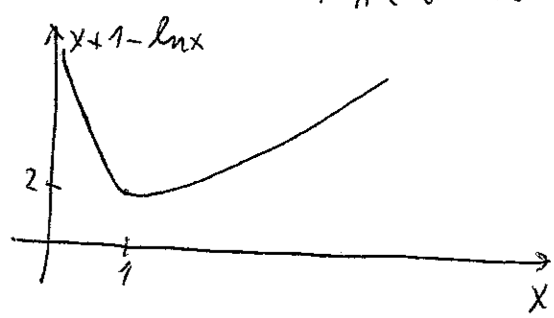
$x+1 \overset{?}{>} \ln x \iff x+1-\ln x \overset{?}{>} 0$

IT IS POSITIVE FOR VALUES CLOSE TO 0

IT'S DERIVATIVE IS $1 - \frac{1}{x}$ WHICH HAS

A STATIONARY POINT IN $x=1$

$\lim_{x \to +\infty} x+1-\ln x = +\infty$. THEREFORE



IT IS ALWAYS POSITIVE

$\Downarrow$

L_2 FSD L_1

$\square$

11.21)

$L_1 \sim LN(\mu_1, \theta^2) \quad L_2 \sim LN(\mu_2, \theta^2) \qquad \mu_1 > \mu_2$

IS THERE FSD OR SSD?

$$F_1 = \phi\left(\frac{\ln x - \mu_1}{\sigma}\right) \qquad F_2 = \phi\left(\frac{\ln x - \mu_2}{\sigma}\right)$$

SINCE ϕ IS INCREASING, WE JUST CHECK

$$\frac{\ln x - \mu_1}{\sigma} \qquad \frac{\ln x - \mu_2}{\sigma} \qquad \forall x \in (0, +\infty)$$

$\theta > 0$ $\quad \ln x - \mu_1$ WITH $\ln x - \mu_1$

$$\ln x - \mu_1 < \ln x - \mu_2 \quad \forall x \in (0, +\infty)$$

$$F_1 < F_2 \quad \forall x \in (0, +\infty)$$

L_1 FSD L_2

$\square$