12.1 VALUE AT RISK

<u>VALUE AT RISK VaR</u> IS THE MAX AMOUNT YOU MAY LOSE WITH A CERTAIN PROBABILITY.

V_0 INITIAL PORTFOLIO VALUE
V_1 FINAL PORTFOLIO VALUE

$$P\left(V_1 - V_0 \leq -VaR\right) = c$$

WHERE $1-c$ IS THE PROBABILITY TO LOSE NOT MORE THAN VaR

$VaR(c)$ IS THE c-LEVEL VALUE AT RISK

BE G THE PRESENT VALUE OF A GAIN FROM AN INVESTMENT.

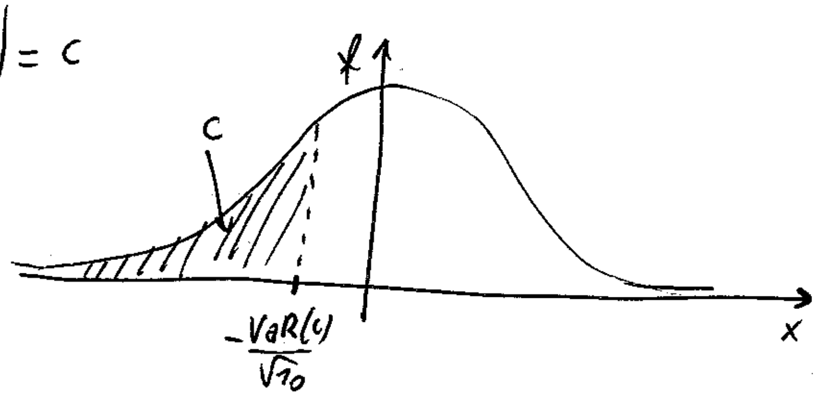
$$P\left(G < -VaR(c)\right) = c$$ DETERMINATES VaR AS FUNCTION OF c

<u>EXAMPLE</u>: $G \sim N(0, \sigma^2)$

$$P\left(G < -VaR(c)\right) = \phi\left(\frac{-VaR(c) - 0}{\sigma}\right) = \phi\left(-\frac{VaR(c)}{\sigma}\right)$$

FOR EXAMPLE, IF $\sigma^2 = 10$ AND IF WE WANT $c = 10\%$

$$\phi\left(-\frac{VaR(c)}{\sqrt{10}}\right) = c$$

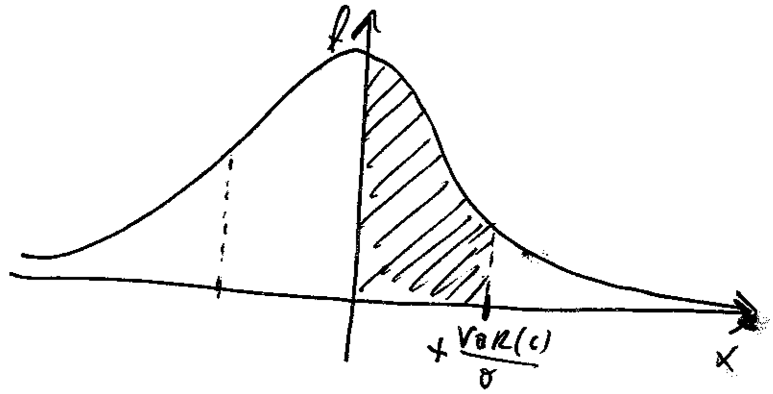


THIS AREA CAN BE FOUND ON TABLES. LOOK FOR AN AREA OF 0,1 AND YOU WILL FIND A VALUE OF -1,28

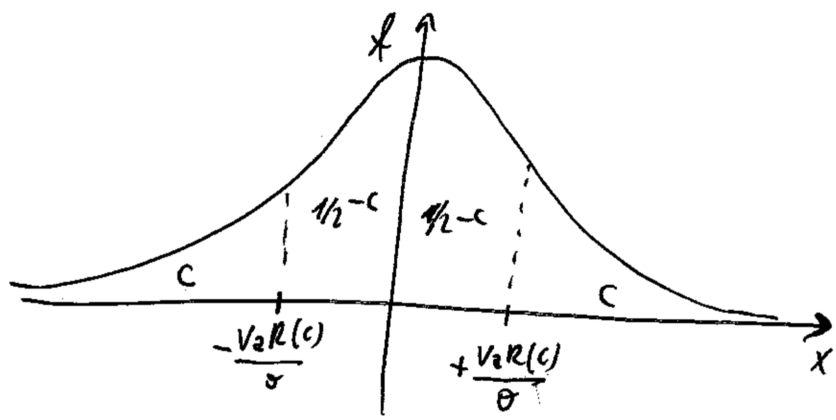
12.2

$$-\frac{VaR(c)}{\sqrt{10}} = -1{,}28 \qquad VaR(c) \approx 4{,}05 \quad \text{FOR} \quad c = 10\%$$

NOTE HOWEVER THAT OFTEN TABLES GIVE THIS AREA



AND TO FIND THE CORRECT VALUES IN THESE TABLES, YOU SHOULD LOOK FOR AN AREA WITH VALUE $\frac{1}{2} - c$ (SINCE GAUSSIAN IS SYMMETRIC) AND REVERSE THE RESULT'S SIGN.



<u>C-LEVEL CONDITIONAL VALUE AT RISK</u> CVAR(C) IS

$$-\frac{1}{c}\int_{-\infty}^{-VaR(c)} s \cdot f(s)\, ds$$

IT IS BASICALLY THE EXPECTED VALUE OF THE GAIN, BUT ONLY FROM $-\infty$ TO $-VaR(c)$. $\frac{1}{c}$ IS A SCALE FACTOR, BECAUSE WHEN C IS VERY SMALL $-VaR(c)$ BECOMES <u>VERY</u> SMALL AND THE INTEGRAL WOULD BE, WITHOUT $\frac{1}{c}$, EXTREMELY SMALL.

NOTE THAT IT WORKS ONLY FOR CONTINOUS R.V.

12.3

ALTERNATIVE EXPRESSION FOR CVAR

INDICATOR FUNCTION: $\chi_{\{y<\alpha\}} = \begin{cases} 1 & \text{WHEN } y<\alpha \\ 0 & \text{WHEN } y \geq \alpha \end{cases}$

THEREFORE

$$CVAR(c) = -\frac{1}{c}\int_{-\infty}^{-VaR(c)} s \cdot f(s)\, ds = -\frac{1}{c}\int_{-\infty}^{+\infty} s \cdot f(s) \cdot \chi_{\{s<-VaR(c)\}}\, ds =$$

$$= -\frac{1}{c} E\left(G \cdot \chi_{\{G<-VaR(c)\}}\right) = -\frac{E\left(G \cdot \chi_{\{G<-VaR(c)\}}\right)}{P\left(G<-VaR(c)\right)}$$

THIS FORMULA WORKS ALSO FOR DISCRETE R.V.

ANOTHER ALTERNATIVE EXPRESSION FOR CVAR

$F_G\left(x \mid G<-VaR(c)\right) = P\left(G<x \mid G<-VaR(c)\right)$: IS THE CONDITIONAL CUMULATIVE DISTRIBUTION FUNCTION

$$= P\left((G<x) \cap (G<-VaR(c))\right) / P\left(G<-VaR(c)\right) =$$

$$= \frac{P\left(G < \min(x, -VaR(c))\right)}{P\left(G<-VaR(c)\right)}$$

$$F_G\left(x \mid G<-VaR(c)\right) = \begin{cases} x \leq -VaR(c) & P(G<x)/P(G<-VaR(c)) \\ x \geq -VaR(c) & P(G<-VaR(c))/P(G<-VaR(c)) = 1 \end{cases}$$

THE CORRESPONDING DENSITY FUNCTION

$$f_G\left(x \mid G<-VaR(c)\right) = \begin{cases} x \leq -VaR(c) & f(x)/P(G<-VaR(c)) \\ x > -VaR(c) & 0 \end{cases}$$

12.4)

$$CVaR(c) = -\frac{1}{c}\int_{-\infty}^{-VaR(c)} s \cdot f(s)\,ds = -\int_{-\infty}^{-VaR(c)} s \cdot \frac{f(s)}{P(G<-VaR)}\,ds =$$

$$= -\int_{-\infty}^{+\infty} s \cdot f_G(s \mid G<-VaR)\,ds = -E\big(G \mid G<-VaR(c)\big)$$

THIS FORMULA WORKS ALSO FOR DISCRETE R.V.

EXAMPLE WITH THE NORMAL DISTRIBUTION

$G \sim N(\mu, \sigma^2)$ FIND $CVaR(c)$ AS A FUNCTION OF c, μ, σ^2, VaR

$$c = \int_{-\infty}^{-VaR(c)} f_N(s)\,ds = \phi\left(\frac{-VaR(c)-\mu}{\sigma}\right)$$ FROM HERE VaR CAN BE CALCULATED

$$CVaR(c) = -\frac{1}{c}\int_{-\infty}^{-VaR} s \cdot \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(s-\mu)^2}{2\sigma^2}}\,ds = -\frac{1}{c}\int_{-\infty}^{-VaR} (s-\mu)\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(s-\mu)^2}{2\sigma^2}}\,ds +$$

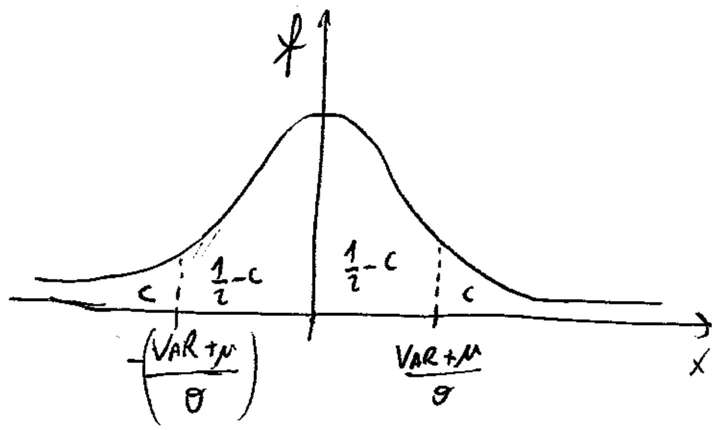
$$-\frac{1}{c}\int_{-\infty}^{-VaR} \mu \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(s-\mu)^2}{2\sigma^2}}\,ds = \left[-\frac{1}{c}\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(s-\mu)^2}{2\sigma^2}} \cdot \left(-\frac{2\sigma^2}{2}\right)\right]_{-\infty}^{-VaR} +$$

$$-\frac{1}{c}\int_{-\infty}^{-VaR} \mu f_{N(\mu,\sigma^2)}(s)\,ds = +\frac{1}{c}\frac{\sigma}{\sqrt{2\pi}}\, e^{-\frac{(-VaR-\mu)^2}{2\sigma^2}} - 0 - \frac{\mu}{c}\cdot c =$$

$$= -\mu + \frac{\sigma}{c\sqrt{2\pi}}\, e^{-\frac{(VaR+\mu)^2}{2\sigma^2}}$$

THIS FORMULA IS VALID FOR ANY $G \sim N(\mu, \sigma^2)$

12.5) $G \sim N(\mu, \sigma^2)$ $\quad c = 0{,}01 \quad$ FIND CVAR AS FUNCTION OF μ, σ^2



$$0.01 = \phi\left(\frac{-VaR-\mu}{\sigma}\right)$$

$$\frac{1}{2} - c = \frac{1}{2} - 0{,}01 = 0{,}49$$

$$-\frac{VaR+\mu}{\sigma} = -2{,}33$$

$$VaR(0{,}01) = -\mu + 2{,}33\,\sigma$$

$$CVaR(0{,}01) = -\mu + \frac{\sigma}{0{,}01\sqrt{\pi 2}}\, e^{-\frac{(\mu + 2{,}33\sigma - \mu)^2}{2\sigma^2}} =$$

$$\approx -\mu + 39{,}89\,\sigma\, e^{-\frac{2{,}33^2}{2}} \approx -\mu + 2{,}64\,\sigma$$

IT IS DIFFERENT FROM VAR.

11.6

SUPPOSE $G = \Delta V$ is $U(-5;5)$
FIND $VAR(0.1)$ AND $CVAR(0.1)$

$$0{,}1 = \int_{-\infty}^{-VAR(0,1)} f_V(s)\,ds = \int_{-5}^{-VAR(0,1)} \frac{1}{10}\,ds = \left[\frac{s}{10}\right]_{-5}^{-VAR(0,1)} = \frac{-VAR(0,1)+5}{10}$$

$$1 = -VAR(0,1) + 5 \qquad VAR(0,1) = 4$$

$$CVAR(0,1) = -\frac{1}{0{,}1}\int_{-\infty}^{-VAR(0,1)} s \cdot f_V(s)\,ds = -\frac{1}{0{,}1}\int_{-5}^{-4} s \cdot \frac{1}{10}\,ds = -\left[\frac{s^2}{2}\right]_{-5}^{-4} = -\frac{16-25}{2} =$$

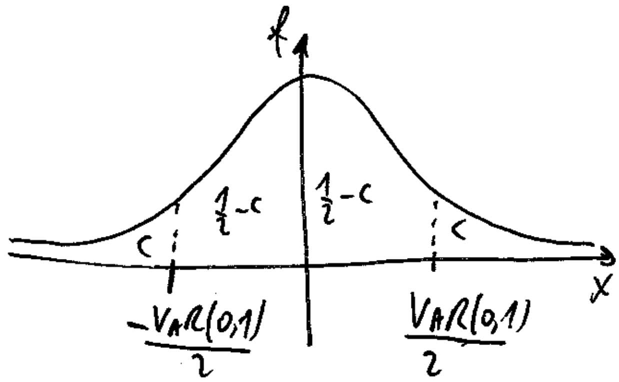
$$= +4{,}5$$

D

12.7 SUPPOSE $\sigma = \Delta V$ IS $N(0;2)$ $\mu = 0$ $\sigma^2 = 2$

FIND $VAR(0.1)$ AND $CVAR(0,1)$

$$0,1 = \int_{-\infty}^{-VAR(0,1)} \frac{1}{\sqrt{2\pi 2}} e^{-\frac{s^2}{2\cdot 2}} ds = \phi\left(\frac{-VAR(0.1) - 0}{\sqrt{2}}\right)$$



$$\frac{1}{2} - c = \frac{1}{2} - 0,1 = 0,4$$

$$\frac{-VAR(0,1)}{\sqrt{2}} = -1,285$$

$VAR(0,1) = 1,81726$

$$CVAR(0,1) = -\frac{1}{0,1} \int_{-\infty}^{-1,82} s \cdot \frac{1}{\sqrt{2\pi 2}} e^{-\frac{s^2}{2\cdot 2}} ds$$

WE CAN USE THIS FORMULA, WHICH IS QUITE COMPLICATED OR APPLY IMMEDIATELY THE FORMULA FOR THE NORMAL

$$CVAR(C) = -\mu + \frac{\sigma}{c\sqrt{2\pi}} e^{-\frac{(\mu + VAR(c))^2}{2\sigma^2}}$$

THE FORMULA LEADS TO $CVAR(0,1) = -0 + \frac{\sqrt{2}}{0,1 \cdot \sqrt{2\pi}} e^{-\frac{1,82^2}{4}} = 2,4716$

ON THE OTHER HAND, DOING ALL THE CALCULATIONS

$$CVAR(0,1) = -2,82 \int_{-\infty}^{-1,82} s \cdot e^{-\frac{s^2}{4}} ds = +2,82 \left[e^{-\frac{s^2}{4}} \cdot 2 \right]_{-\infty}^{-1,82} = 5,64 \cdot \left(e^{-\frac{1,82^2}{4}} - 0 \right) =$$

$= 2,4702$

□

$\boxed{12.8}$ $CVAR(0.1) = 3x + 2y$ WHERE (x, y) MAY BE

$(1, 2)$ $(2, 1)$ $(0.5, 5)$ $(0.25, 3)$

WHICH INVESTMENT IS THE BEST ?

CVAR IS THE EXPECTED VALUE OF THE LOSS FROM $-\infty$ TO $VAR(0.1)$.

THEREFORE WE WOULD LIKE THAT IT BE AS SMALL AS POSSIBLE

$(1, 2) \longrightarrow 3 + 4 = 7$
$(2, 1) \longrightarrow 6 + 2 \rightarrow 8$
$(0.5, 5) \longrightarrow 1.5 + 10 = 11.5$
$(0.25, 3) \longrightarrow 0.75 + 6 = 6.75$ $\longrightarrow$ THIS IS THE BEST INVESTMENT

SO THE BEST INVESTMENT IS ALWAYS THE ONE WITH THE LOWEST VAR OR CVAR (WHEN C IS THE SAME)

$\square$