13.1 DIVERSIFICATION AND RISK-SPREADING

SUPPOSE YOU CAN BUY TWO STOCKS WITH A PRICE OF 10€

- A SUNGLASSES COMPANY WHICH CAN GO FROM 5 € TO 20€ DEPENDING ON HOW SUNNY THE SUMMER IS (UNIFORM DISTR.)
- AN UMBRELLA COMPANY WHICH CAN GO FROM 5 € TO 20 € DEPENDING ON HOW MUCH RAINY THE SUMMER IS (UNIFORM DISTR.)

IF YOU INVEST ALL ON THE SUNGLASSES COMPANY YOUR RATE OF RETURN IS UNIFORM FROM −50% TO +100%

THE SAME IF YOU INVEST EVERYTHING IN THE UMBRELLA COMPANY.

IN BOTH CASES THE E(RATE OF RETURN) = +25%
WITH VALUES RANGING FROM −50% TO +100%

IF YOU INSTEAD INVEST HALF YOUR MONEY ON SUNGLASSES AND HALF ON UMBRELLAS, CONSIDERING THAT "RAINY" AND "SUNNY" ARE PERFECTLY NEGATIVELY CORRELATED, THE RATE OF RETURN IS

$$0{,}5\left[-50\% \cdot (P(\text{RAINY})) + 100\% \cdot (P(\text{SUNNY}))\right] + 0{,}5\left[-50\% \cdot (P(\text{SUNNY})) + 100\% \cdot (P(\text{RAINY}))\right] =$$

$$= 0{,}5 \cdot (-50\% + 100\%) \cdot (P(\text{RAINY})) + 0{,}5 \cdot (-50\% + 100\%) \cdot (P(\text{SUNNY})) =$$

$$= 0{,}5 \cdot (+50\%) \cdot (P(\text{RAINY}) + P(\text{SUNNY})) = +25\% \cdot (1) = +25\%$$

IT IS <u>COSTANTLY</u> +25% $\quad E(\text{RATE OF RETURN}) = +25\%$

<u>THE SAME</u> EXPECTED RATE OF RETURN BUT WITH <u>MUCH LESS RISK</u>

13.2] WHAT IF THE TWO STOCKS ARE INDIPENDENT?

TO SEE THE DISTRIBUTION OF RATE OF RETURN WE SIMPLIFY

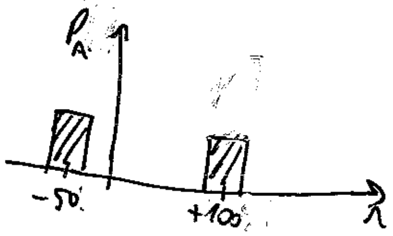
STOCK A $r_A = (-50\% ; 50\% ; +100\% , 50\%)$ $E(r_A) = +25\%$ $VAR(r_A) = 0{,}5625$

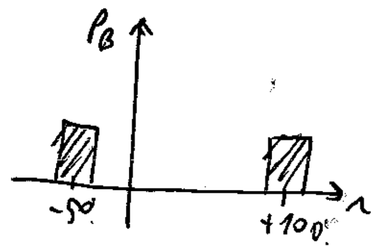
STOCK B $r_B = (-50\% , 50\% ; +100\% , 50\%)$ $E(r_B) = +25\%$ $VAR(r_B) = 0{,}5625$

$r = 0{,}5\, r_A + 0{,}5\, r_B$ GOES FROM -50 TO $+100$

$E(r) = 0{,}5\, E(r_A) + 0{,}5\, E(r_B) = +25\%$

BUT THE DISTRIBUTION





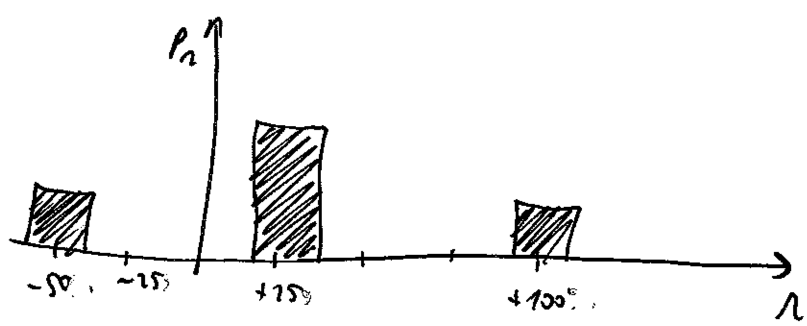
A -50% B -50% $\rightarrow$ $r = -50\%$

A -50% B $+100\%$ $\rightarrow$ $r = +25\%$

A $+100\%$ B -50% $\rightarrow$ $r = +25\%$

A $+100\%$ B $+100\%$ $\rightarrow$ $r = +100\%$

$$VAR(r) = \left(-\tfrac{1}{2} - \tfrac{1}{4}\right)^2 \cdot 0{,}25 + \left(+\tfrac{1}{4} - \tfrac{1}{4}\right)^2 \cdot 0{,}5 + \left(1 - \tfrac{1}{4}\right)^2 \cdot 0{,}25 = 0{,}28125$$



$E(r)$ IS THE SAME AS $E(r_A)$ AND $E(r_B)$, MIN AND MAX ARE THE SAME, BUT $VAR(r)$ IS HALF OF $VAR(r_A)$ AND $VAR(r_B)$

IN GENERAL, THE VARIANCE <u>DECREASES</u> COMBINING INDIPENDENT INVESTMENTS.

13.3 MEAN VARIANCE APPROACH

USING THIS APPROACH WE SUPPOSE THAT THE INVESTOR IS ONLY INTERESTED IN EXP. VALUE AND VARIANCE OF THE R.O.R. AND NOT IN THE EFFECTIVE DISTRIBUTION.

THEREFORE, HIS UTILITY FUNCTION IS $U = U(\mu, \sigma)$ WHERE μ IS $E(\text{R.O.R.})$ AND $\sigma^2 = VAR(\text{R.O.R.})$

SUPPOSE NOW AS USUAL TO HAVE A RISK-FREE ASSET WITH R.O.R. r AND A RISKY ASSET WITH R.O.R. ξ

$R = E(\xi) \quad \Sigma^2 = VAR(\xi)$

SUPPOSE $r < R$, TO GIVE ALSO TO RISK-AVERSE AGENTS THE POSSIBILITY TO INVEST IN THE RISKY ASSET.

IF WE INVEST A FRACTION X IN THE RISKY ASSET AND A FRACTION $(1-X)$ IN THE RISK-FREE ASSET, THE COMBINED R.O.R. IS $X\xi + (1-X)r$. ITS EXPECTED VALUE

$$\mu_X = E(X\xi + (1-X)r) = X E(\xi) + (1-X)r = XR + (1-X)r = X(R-r) + r$$

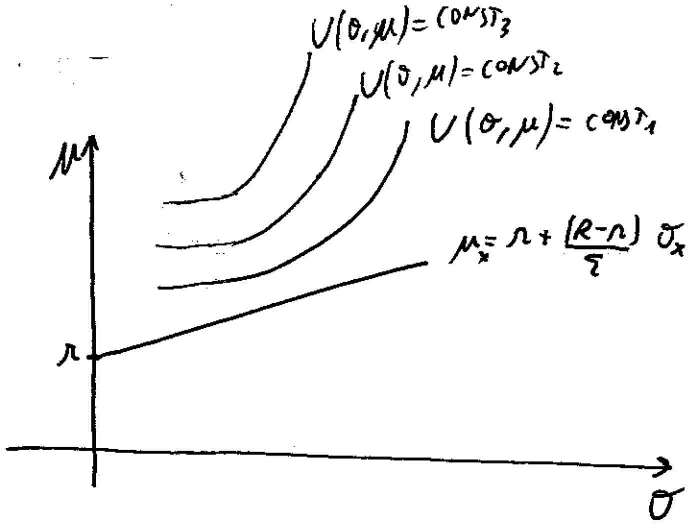
AND ITS VARIANCE

$$\sigma_X^2 = VAR(X\xi + (1-X)r) = VAR(X\xi) = X^2 \, VAR(\xi) = X^2 \Sigma^2$$

NOTE THAT $\mu_X = \frac{\sigma_X}{\Sigma}(R-r) + r$ IS THE RELATION BETWEEN μ_X AND σ_X. THE LARGER σ_X, THE $\frac{R-r}{\Sigma}$ LARGER BECOMES μ_X

13.4)

WE ARE NOW INTERESTED IN STUDYING THE VALUES OF OUR UTILITY FUNCTION WHEN μ AND σ^2 CHANGE WITH THE BOUND THAT $\mu_x = \frac{\sigma_x}{\Sigma}(R-r)+r$



μ_x IS EXP. VALUE OF R.OR. OF COMBINED INV

σ_x IS VARIANCE OF R.OR. OF COMBINED IN

THE LINE $\mu_x = r + \left(\frac{R-r}{\Sigma}\right)\sigma_x$ IS THE <u>CAPITAL MARKET LINE</u>

ITS SLOPE $\frac{R-r}{\Sigma}$ IS THE PRICE FOR RISK, A MEASURE OF HOW IMPROVING THE RISK, σ, AFFECTS THE RETURN, μ.

WE NEED TO MAXIMIZE $U(\sigma,\mu)$ WITH THE COSTRAINT THAT $\mu = r + \frac{(R-r)}{\Sigma}\sigma$, SO FIND A POINT ON THE LINE WHICH MAXIMAZES U

USING LAGRANGIAN FUNCTION $\mathcal{L} = U(\sigma,\mu) - \lambda\left(\mu - r - \frac{(R-r)}{\Sigma}\sigma\right)$

WE GET

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \mu} = 0 \\ \frac{\partial \mathcal{L}}{\partial \sigma} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \end{cases} \qquad \begin{cases} \frac{\partial U}{\partial \mu}(\sigma,\mu) - \lambda = 0 \\ \frac{\partial U}{\partial \sigma}(\sigma,\mu) + \lambda\frac{(R-r)}{\Sigma} = 0 \\ -\mu + r + \frac{R-r}{\Sigma}\sigma = 0 \end{cases}$$

AND FROM THESE EQUATIONS WE FIND (σ^*, μ^*)

13.5

WHAT HAPPENS IF WE USE THE ARROW-PRAAT TO APPROXIMATE U FOR $K\mu$ AND $K^2\theta^2$ WITH K SMALL!

$$C^e \approx \mu K - \frac{1}{2} A(w_0 - K\mu) K^2\theta^2 \approx \mu K - \frac{1}{2} A(w_0) K^2 \theta^2$$

IF WE SAY THAT THE UTILITY OF A LOTTERY CAN BE EXPRESSED BY ITS CERTAINTY EQUIVALENT

$$U(\theta, \mu) \approx \mu K - \frac{1}{2} A(w_0) K^2 \theta^2$$

CALLING $\tilde{\mu} = K\mu$ $\tilde{\theta} = K\theta$

$$U(\tilde{\theta}, \tilde{\mu}) \approx \tilde{\mu} - \frac{1}{2} A(w_0) \tilde{\theta}^2$$

$$\Downarrow$$

"U, CLOSE TO $\tilde{\mu} = 0$ AND $\tilde{\theta} = 0$, INCREASES AS $\tilde{\mu}$ INCREASING, AND FOR A RISK-AVERSE AGENT $(A<0)$ IT DECREASES WHILE $\tilde{\theta}$ DECREASES.

13.6) CAPITAL ASSET PRICING MODEL

$$\mu_x = r + \frac{\sigma_x}{\Sigma}(R-r)$$ CAN BE GENERALIZED WITH

$$\mu_i = r + \beta_i (R-r)$$ WHERE BETA IS A MEASURE OF HOW RISKY IS YOUR INVESTMENT (σ_x) COMPARED TO THE VARIABILITY OF THE MARKET (Σ)

EXAMPLE CURRENT INTEREST RATE 4%

$R = 1{,}1$ AND $\Sigma = 0{,}25$ OF THE RISKY MARKET

WE CHOOSE $X = 30\%$.

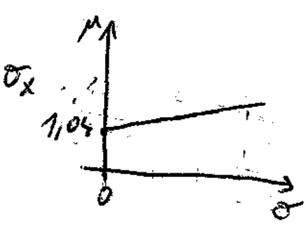
FIND μ_x, σ_x AND β.

$r = 1{,}04$ NON-RISKY ASSET'S RATE OF RETURN

$R = 1{,}1$

$\Sigma = 0{,}25$

$$\mu_x = 1{,}04 + \frac{\sigma_x}{0{,}25}\, 0{,}06 = 1{,}04 + 0{,}24\,\sigma_x$$



$$\sigma_x = X\,\Sigma = 30\% \cdot 0{,}25 = 0{,}075$$

$$\mu_x = 1{,}04 + \frac{0{,}075}{0{,}25}\, 0{,}06 = 1{,}04 + 0{,}3 \cdot 0{,}06 = 1{,}058$$

THIS MEANS THAT MY PORTFOLIO WILL GAIN ON AVERAGE 5,8% WITH A RISK OF 7,5%. β IS $0{,}3 = 30\%$

13.7

WITHIN THE MEAN VARIANCE APPROACH, WOULD A RISK AVERSE AGENT BE INDIFFERENT BETWEEN $L^1 = R(-0.1, 0.1)$ AND $L^2 = N(0, 0.1)$?

$E(L^1) = 0 \qquad E(L^2) = 0$

$VAR(L^1) = \frac{0.2^2}{12} = \frac{0.04}{12} = \frac{0.01}{3}$

$VAR(L^2) = 0.1$

IN MEAN-VARIANCE APPROACH $U = U(E(L), VAR(L))$

THIS UTILITY OBVIOUSLY IN INCREASING WITH RESPECT OF $E(L)$, $\frac{\partial U}{\partial E(L)} > 0$, AND FOR RISK-AVERSE AGENT IS DECREASING WITH RESPECT TO VARIANCE, $\frac{\partial U}{\partial VAR(L)} < 0$.

THEREFORE A RISK AVERSE AGENT WILL HAVE $U(L^2) < U(L^1)$ SINCE $VAR(L^2) > VAR(L^1)$.

AGENT PREFERS L^1.

$\square$

13.8]

WITHIN THE MEAN-VARIANCE APPROACH, WOULD A RISK-AVERSE AGENT BE INDIFFERENT BETWEEN $L^1 \sim R(0;2)$ AND $L^2 \sim N(1;1)$?

WE ARE IN MEAN-VARIANCE APPROCH, THEREFORE WE SIMPLY CONSIDER EXPECTED VALUE AND VARIANCE

$E(L^1) = 1 \qquad E(L^2) = 1$

$VAR(L^1) = \frac{(b-a)^2}{12} = \frac{(2-0)^2}{12} = \frac{4}{12} = \frac{1}{3}$

$VAR(L^2) = 1$

SAME EXPECTED VALUE, VARIANCE OF L^2 IS LARGER, AGENT IS RISK AVERSE.

$\Downarrow$

AGENT PREFERS L^1

$\square$