$\boxed{1.1}$ $P(A) = 0{,}5$ $P(B) = 0{,}3$ $P(A \cap B) = 0{,}2$

FIND $P(A \cup B)$, $P(C(A \cap B))$, $P(A|B)$

HW: FIND $P(C(A \cup B))$, $P(CA \cup CB)$, $P(CA \cap CB)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0{,}5 + 0{,}3 - 0{,}2 = 0{,}6$$

$$P(C(A \cap B)) = 1 - P(A \cap B) = 1 - 0{,}2 = 0{,}8$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0{,}2}{0{,}3} = \frac{2}{3}$$

$$P(C(A \cup B)) = 1 - P(A \cup B) = 1 - 0{,}6 = 0{,}4$$

$$P(CA \cup CB) = P(C(A \cap B)) = 1 - P(A \cap B) = 1 - 0{,}2 = 0{,}8$$

$$P(CA \cap CB) = P(C(A \cup B)) = 1 - P(A \cup B) = 1 - 0{,}6 = 0{,}4$$

$\square$

1.2 $P(A)=0{,}82 \quad P(B)=0{,}38$

ARE EVENTS A AND B MUTUALLY EXCLUSIVE ?

A AND B ARE MUTUALLY EXCLUSIVE IF AND ONLY IF $A \cap B = \emptyset$

SINCE $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Downarrow$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0{,}82 + 0{,}38 - P(A \cup B) =$$

$$= 1{,}2 - P(A \cup B)$$

SINCE $P(A \cup B)$ MAY NEVER BE > 1

$$\Downarrow$$

$$P(A \cap B) = 1{,}2 - P(A \cup B) \geq 0{,}2$$

$$\Downarrow$$

$$P(A \cap B) > 0$$

$$\Downarrow$$

$$A \cap B \neq \emptyset$$

A AND B ARE NOT MUTUALLY EXCLUSIVE

TWO EVENTS TO BE MUTUALLY EXCLUSIVE MUST AT LEAST HAVE $P(A) + P(B) \leq 1$. HOWEVER, THIS IS ONLY A NECESSARY CONDITION BUT NOT SUFFICIENT. IN FACT, WE MAY TAKE A DIE AND SAY $A = \{1,2,3\} \quad B = \{3,4\}$

$P(A) = \frac{3}{6} \quad P(B) = \frac{2}{6} \quad P(A) + P(B) = \frac{5}{6} < 1$

BUT $A \cap B = \{3\}$ AND THEY STILL ARE NOT MUTUALLY EXCLUSIVE

$\square$

(1.3) CAN TWO MUTUALLY EXCLUSIVE EVENTS BE INDEPENDENT?

$A \cap B = \emptyset$

CAN $P(A \cap B) = P(A) \cdot P(B)$?

SINCE $P(A \cap B) = P(\emptyset) = 0$

A AND B MAY BE INDEPENDENT IF AND ONLY IF $P(A) = 0$ <u>OR</u> $P(B) = 0$

$\square$

(1.4) E_1 IS A "HEAD" RESULT OF A COIN FLIP

E_2 IS 3 OR 6 ON A DIE

EVALUATE $P(\overset{HW}{C}E_1)$ $\quad P(CE_2)$ $\quad P(E_1 \cap E_2)$

$P(E_1 \cap CE_2)$ $\quad P(E_1 \mid E_2)$ $\quad P(CE_1 \overset{HW}{\cup} CE_2)$

$P(E_1) = 0{,}5 \qquad P(E_2) = \frac{1}{3}$

- $P(CE_1) = 1 - 0{,}5 = 0{,}5$
- $P(CE_2) = 1 - \frac{1}{3} = \frac{2}{3}$
- $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2) = 0{,}5 \cdot \frac{1}{3} = \frac{1}{6}$

  THE TWO EVENTS ARE INDEPENDENT

- $P(E_1 \cap CE_2) = P(E_1) \cdot P(CE_2) = 0{,}5 \cdot \frac{2}{3} = \frac{1}{3}$
- $P(E_1 \mid E_2) = \dfrac{P(E_1 \cap E_2)}{P(E_2)} = \dfrac{P(E_1) \cdot P(E_2)}{P(E_2)} = P(E_1) = 0{,}5$

  OBVIOUS SINCE E_1 AND E_2 ARE INDEPENDENT

- $P(CE_1 \cup CE_2) = P(CE_1) + P(CE_2) - P(CE_1 \cap CE_2) = 0{,}5 + \frac{2}{3} - 0{,}5 \cdot \frac{2}{3} = \dfrac{3 + 4 - 2}{6} = \dfrac{5}{6}$

OR

$P(CE_1 \cup CE_2) = P(C(E_1 \cap E_2)) = 1 - P(E_1 \cap E_2) = 1 - 0{,}5 \cdot \frac{1}{3} = \dfrac{6-1}{6} = \dfrac{5}{6}$

$\square$

1.5 A MACHINE IS INCORRECTLY ADJUSTED 10% OF TIME. WHEN INCORRECT IT PRODUCES 30% OF WRONG PIECES. WHEN CORRECT IT PRODUCES 10% OF WRONG PIECES.

AN ISPECTOR TAKES RANDOMLY A WRONG PIECE.
WHAT IS THE PROBABILITY THAT THE MACHINE IS INCORRECTLY ADJUSTED?

$$P(\text{INCORRECTLY ADJ} \mid \text{EXTRACTED WRONG PIECE}) =$$

$$= \frac{P(\text{INCORR ADJ} \cap \text{EXTRACT WRONG PIECE})}{P(\text{EXTRACT WRONG PIECE})} =$$

INCORR ADJ 10%
- WRONG 3%
- RIGHT 7%

CORR ADJ 90%
- WRONG 9%
- RIGHT 81%

$$= \frac{0{,}03}{0{,}12} = \frac{3}{12} = \frac{1}{4} = 25\%$$

PAY ATTENTION: IT IS $\underline{\text{NOT}}$ $P(\text{EXTRACT WRONG P} \mid \text{INCORR ADJ}) =$
$= 33\%$

AND IT IS $\underline{\text{NOT}}$ $P_D(\text{WRONG P}) = 12\%$

1.6

A DIE IS ROLLED TWICE.
WHAT IS THE PROBABILITY THAT THE SUM BE 7 ? WHAT IS THIS PROB. <u>IF YOU KNOW</u> THAT THE SUM IS ODD?

LET'S ANALYZE ALL CASES. 7 CAN BE OBTAINED WITH

1,6 ; 2,5 ; 3,4 ; 4,3 ; 5,2 ; 6,1

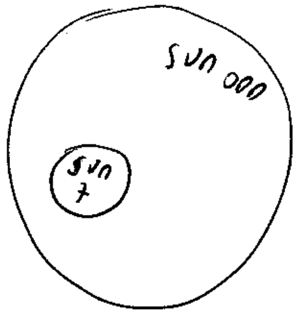
1/6 1/6 1/6 1/6 1/6 1/6 1/6 1/6 1/6 1/6 1/6 1/6

SINCE THE TWO TOSSING ARE INDEPENDENT

$$P(1,6) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} \qquad P(\text{SUM}=7) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{6}{36} = \frac{1}{6}$$

$$P(\text{SUM IS } 7 \mid \text{SUM IS ODD}) = \frac{P((\text{SUM IS } 7) \cap (\text{SUM IS ODD}))}{P(\text{SUM IS ODD})} =$$

$$= \frac{P(\cancel{\text{SUM IS}}\ 7)}{P(\text{SUM IS ODD})} =$$



$$= \frac{1/6}{P(\text{SUM IS ODD})}$$

SUM IS ODD WITH THESE RESULTS:

1,2 ; 1,4 ; 1,6 ; 2,1 ; 2,3 ; 2,5 ; 3,2 ; 3,4 ; 3,6 ; 4,1 ; 4,3 ; 4,5 ;
5,2 ; 5,4 ; 5,6 ; 6,1 ; 6,3 ; 6,5

$$\frac{18}{36} = \frac{1}{2}$$

$$P(\text{SUM IS } 7 \mid \text{SUM IS ODD}) = \frac{1/6}{1/2} = \frac{1}{3} \qquad \square$$

1.7) A SYSTEM ANALYST IS CONCERNED ABOUT PROBAB. OF WRONG RECORDS IN THE INVENTORY SYSTEM.

PROPORTION OF ERRORS	PROBABILITY
0,10	50%
0,15	30%
0,20	20%

A RANDOM SAMPLE OF 25 RECORDS YIELDED 3 INCORRECT RECORDS

FIND THE NEW PROBABILITIES FOR THE PROPORTIONS OF ERRORS, TAKING INTO ACCOUNT THE 3 ERRORS OUT OF 25.

$$P(\text{PROP IS } 0{,}10 \mid 3 \text{ OUT } 25) = \frac{P((\text{PROP IS } 0{,}10) \cap (3 \text{ OUT } 25))}{P(3 \text{ OUT } 25)}$$

UNFORTUNATELY, WE CANNOT CALCULATE

$P((\text{PROP IS } 0{,}10) \cap (3 \text{ OUT } 25))$ BECAUSE THE EVENTS ARE NOT INDEPENDENT

BUT WE KNOW THAT

$$P(3 \text{ OUT } 25 \mid \text{PROP IS } 0{,}10) = \frac{P((3 \text{ OUT } 25) \cap (\text{PROP IS } 0{,}10))}{P(\text{PROP IS } 0{,}10)}$$

$$\Downarrow$$

$$P((3 \text{ OUT } 25) \cap (\text{PROP IS } 0{,}10)) = P(3 \text{ OUT } 25 \mid \text{PROP IS } 0{,}10) \cdot P(\text{PROP IS } 0{,}10)$$

$$P(\text{PROP IS } 0{,}10 \mid 3 \text{ OUT } 25) = \frac{P(3 \text{ OUT } 25 \mid \text{PROP IS } 0{,}10) \cdot P(\text{PROP IS } 0{,}10)}{P(3 \text{ OUT } 25)}$$

1. 8

$P(\text{PROP is } 0{,}10) = 50\%$

$P(3 \text{ OUT } 25 \mid \text{PROP is } 0{,}10)$ is a BINOMIAL $= \binom{25}{3} 0{,}1^3\, 0{,}9^{22} =$

$$= \frac{25!}{3! \cdot 22!}\, 0{,}1^3 \cdot 0{,}9^{22} = 2300 \cdot 0{,}1^3 \cdot 0{,}9^{22} = 0{,}2265$$

TO CALCULATE $P(3 \text{ OUT } 25)$ WE NEED:

$P(3 \text{ OUT } 25) = P(3 \text{ OUT } 25 \mid \text{PROP. is } 0{,}1) \cdot 0{,}5 + P(3 \text{ OUT } 25 \mid \text{PROP is } 0{,}15) \cdot 0{,}3$
$+ P(3 \text{ OUT } 25 \mid \text{PROP is } 0{,}2) \cdot 0{,}2$

$P(3 \text{ OUT } 25 \mid \text{PROP is } 0{,}15) = \binom{25}{3} 0{,}15^3\, 0{,}85^{22} = 2300 \cdot 0{,}15^3 \cdot 0{,}85^{22} = 0{,}2174$

$P(3 \text{ OUT } 25 \mid \text{PROP is } 0{,}2) = 0{,}1358$

$P(3 \text{ OUT } 25) = 0{,}2056$

$$P(\text{PROP is } 0{,}10 \mid 3 \text{ OUT } 25) = \frac{0{,}2265 \cdot 0{,}5}{0{,}2056} = 0{,}5508$$

$$P(\text{PROP is } 0{,}15 \mid 3 \text{ OUT } 25) = \frac{0{,}2174 \cdot 0{,}3}{0{,}2056} = 0{,}3172$$

$P(\text{PROP is } 0{,}2 \mid 3 \text{ OUT } 25) = 1 - 0{,}5508 - 0{,}3172 = 0{,}1320$

D

1.9A) GIVEN $f(x) = \begin{cases} Cxe^{-x^2} & x > 0 \\ 0 & x < 0 \end{cases}$ FIND C SUCH THAT $f(x)$ IS A DENSITY FUNCTION

CALCULATE MEDIAN OF THIS R.V. AND THE MODE

(1) $f(x) \geq 0 \quad \forall x$ $\qquad Cx e^{-x^2} \geq 0 \quad \forall x > 0$

$$x \to \text{pos} \qquad e^{-x^2} \to \text{pos}$$

$$\Downarrow$$

$$C \geq 0$$

(2) $\int_{-\infty}^{+\infty} f(s)\,ds = 1$

$$\int_{-\infty}^{+\infty} f(s)\,ds = \int_{-\infty}^{0} 0\,ds + \int_{0}^{+\infty} Cse^{-s^2}\,ds = 0 + C\int_{0}^{+\infty} se^{-s^2}\,ds =$$

= BY SUBSTITUTION $s^2 = t \qquad \frac{dt}{ds} = 2s$

$$dt = 2s\,ds \qquad \frac{dt}{2s} = ds$$

$$= C\int_{0}^{+\infty} \cancel{s}\, e^{-t}\,\frac{dt}{2\cancel{s}} = \frac{C}{2}\int_{0}^{+\infty} e^{-t}\,dt = \frac{C}{2}\left[-e^{-t}\right]_{0}^{+\infty} =$$

$$= \frac{C}{2}\left(0 + e^{-0}\right) = \frac{C}{2}\cdot 1 = \frac{C}{2}$$

$$\frac{C}{2} = 1 \qquad \boxed{C = 2}$$

□

1.9 B)

MEDIAN

$$\frac{1}{2} = \int_{-\infty}^{\eta} f(s)\,ds \qquad \frac{1}{2} = \int_{-\infty}^{0} 0\,ds + \int_{0}^{\eta} C s e^{-s^2} ds$$

SAME SUBSTITUTION AS BEFORE, BUT PAY ATTENTION TO EXTREMES!

$$\cancel{\frac{1}{2}} = 0 + \frac{C}{2}\int_{0}^{\eta^2} e^{-t}\,dt = \frac{C}{\cancel{2}}\left[-e^{-t}\right]_{0}^{\eta^2}$$

$$1 = C\left(-e^{-\eta^2}+1\right) \qquad -\frac{1}{C}+1 = e^{-\eta^2} \qquad \ln\left(1-\frac{1}{C}\right) = -\eta^2$$

$$\eta^2 = -\ln\left(1-\frac{1}{C}\right) \qquad \eta = \pm\sqrt{-\ln\left(1-\frac{1}{C}\right)}$$

I REMOVE THE MINUS BECAUSE SINCE $f(x)$ FOR $x<0$ IS ALWAYS ZERO THE MEDIAN CANNOT BE NEGATIVE

$$\eta = +\sqrt{-\ln\left(1-\frac{1}{C}\right)} = +\sqrt{-\ln\frac{1}{2}} = +\sqrt{\ln 2}$$

MODE TO CALCULATE IT, WE NEED TO FIND THE MAXIMUM OF $f(x)$

$$f'(x) = C\,1\,e^{-x^2} + Cx\,e^{-x^2}\cdot(-2x) = Ce^{-x^2}\left[1-2x^2\right]$$

$$f'(x)=0 \Rightarrow 1-2x^2=0 \qquad 1=2x^2 \qquad x=\pm\sqrt{\frac{1}{2}}$$

AS BEFORE $x=-\sqrt{\frac{1}{2}}$ IS NOT THE MODE BECAUSE FOR $x<0$ $f(x)=0$ AND THIS CANNOT BE A LOCAL MAXIMUM

IF WE WANT TO BE VERY SURE THAT IT IS A MAXIMUM WE SHOULD CHECK THAT $f''\left(+\sqrt{\frac{1}{2}}\right)<0$

D