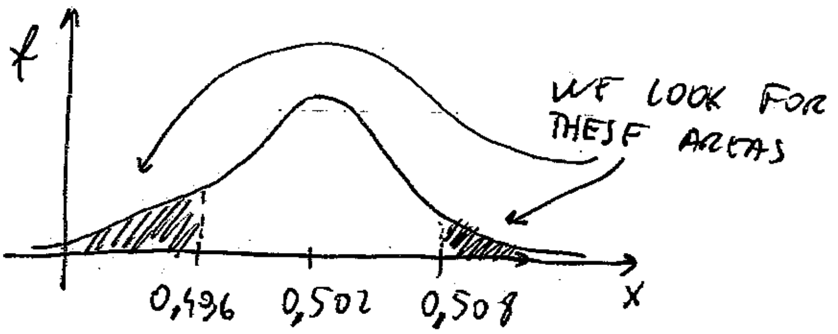
3.1

A MACHINE PRODUCES PIECES WHOSE LENGTH IS <u>NORMALLY</u> DISTRIBUTED WITH $\mu = 0{,}502$ CM AND $\sigma^2 = 0{,}005^2$ CM^2

WRONG PIECES ARE THOSE SHORTER THAN 0,496 CM OR LONGER THAN 0,508 CM

WHAT IS THE PROBABILITY OF PRODUCING A WRONG PIECE?



$$P(\text{WRONG PIECE}) = 1 - \int_{0{,}496}^{0{,}508} \frac{1}{\sqrt{2\pi\, 0{,}005^2}}\, e^{-\frac{(s-0{,}502)^2}{\sqrt{2\cdot 0{,}005^2}}}\, ds =$$

$$= 1 - P\left(N(0{,}502 ; 0{,}005^2) \in [0{,}496 ; 0{,}508]\right) =$$

WE STANDARDIZE

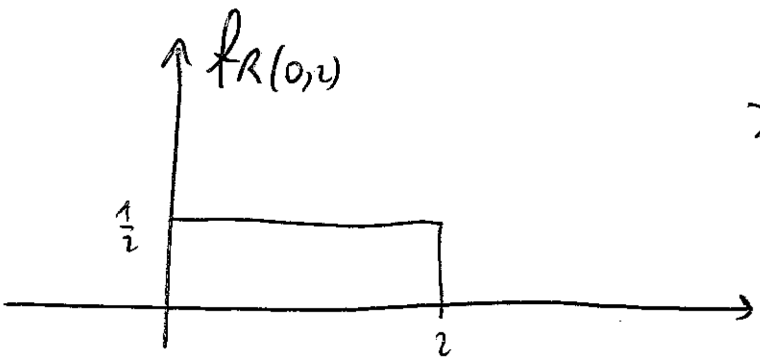
$$= 1 - P\left(N(0;1) \in \left[\frac{0{,}496 - 0{,}502}{0{,}005} ; \frac{0{,}508 - 0{,}502}{0{,}005}\right]\right) =$$

$$= 1 - P\left(N(0,1) \in [-1{,}2 ; 1{,}2]\right) = 1 - (0{,}3849 \cdot 2) \approx 23\%$$

$\square$

3.2 X_c IS THE INDICATOR THAT A UNIFORM RANDOM VARIABLE OVER $[0,2]$ IS LARGER THAN 0,75

CHARACTERIZE X_c



$$X_c = \begin{cases} 0 & X \leq 0{,}75 \\ 1 & X > 0{,}75 \end{cases}$$

$$P(X > 0{,}75) = \int_{0{,}75}^{2} \frac{1}{2}\, ds = \left[\frac{s}{2}\right]_{0{,}75}^{2} = \frac{2}{2} - \frac{0{,}75}{2} = \frac{1{,}25}{2} = 0{,}625$$

$$X_c = \begin{cases} 0 & = 1 - 0{,}625 \\ 1 & = 0{,}625 \end{cases}$$

A BERNOULLI RANDOM VARIABLE WITH $p = 0{,}625$

D

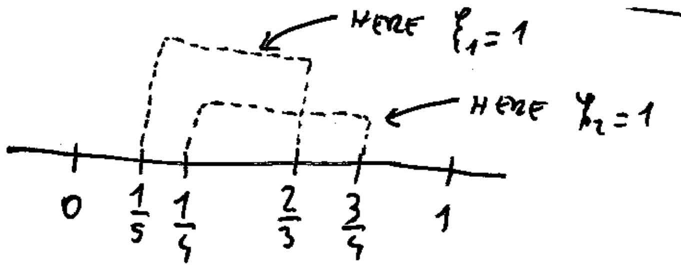
3.3 A)

$\xi \sim R(0,1)$ $\qquad \xi_i = \chi_{\xi \in (a_i, b_i)}$

WHERE $a_1 = \frac{1}{5}$ $\quad b_1 = \frac{2}{3}$

$a_2 = \frac{1}{4}$ $\quad b_2 = \frac{3}{4}$

FIND:
- JOINT DISTRIBUTION OF ξ_1 AND ξ_2
- MARGINAL DISTRIBUTION OF ξ_1 AND ξ_2
- $\mathrm{CORR}(\xi_1, \xi_2)$
- ARE ξ_1 AND ξ_2 INDEPENDENT?



MARGINAL DISTRIBUTION OF ξ_1

$$\xi_1 = \begin{cases} 0 & \xi \le \frac{1}{5} \\ 1 & \xi \in \left(\frac{1}{5}, \frac{2}{3}\right) \\ 0 & \xi \ge \frac{2}{3} \end{cases}$$

$$P\left(\xi \le \frac{1}{5}\right) = \int_0^{\frac{1}{5}} 1\, ds = \frac{1}{5} \qquad P\left(\xi \ge \frac{2}{3}\right) = \int_{\frac{2}{3}}^{1} 1\, ds = \frac{1}{3}$$

THEREFORE

$$\xi_1 = \begin{cases} 0 & \text{PROB} = \frac{1}{5} + \frac{1}{3} = \frac{8}{15} \\ 1 & \text{PROB} = \frac{7}{15} \end{cases}$$

MARGINAL PROBABILITIES FOR ξ_1 ARE $P(0) = \frac{8}{15}$ $\quad P(1) = \frac{7}{15}$

3.3 B)

$$\xi_2 = \begin{cases} 0 & \xi \le \frac{1}{4} \\ 1 & \xi \in \left(\frac{1}{4}, \frac{3}{4}\right) \\ 0 & \xi \geq \frac{3}{4} \end{cases}$$

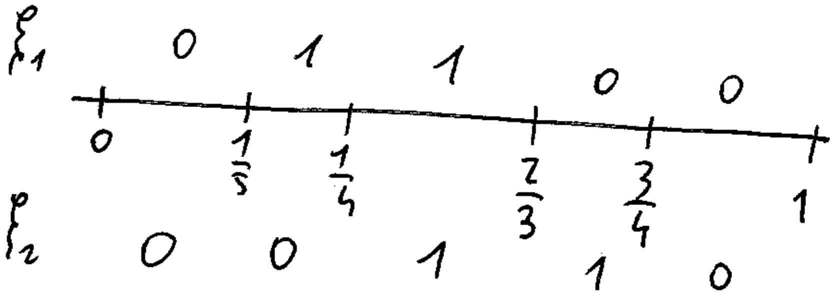
$P\left(\xi \le \frac{1}{4}\right) = \frac{1}{4}$ $\quad P\left(\xi \geq \frac{3}{4}\right) = \frac{1}{4}$

$$\xi_2 = \begin{cases} 0 & \text{PROB} = \frac{1}{2} \\ 1 & \text{PROB} = \frac{1}{2} \end{cases}$$

MARGINAL PROBABILITIES ARE

$P(0) = \frac{1}{2}$ $\quad P(1) = \frac{1}{2}$

JOINT DISTRIBUTION



$(\xi_1, \xi_2) = (0,0)$ WHEN $\xi \le \frac{1}{5}$ OR $\xi \geq \frac{3}{4}$ $\quad$ PROB $= \frac{1}{5} + \frac{1}{4} = \frac{9}{20}$

$(\xi_1, \xi_2) = (1,0)$ WHEN $\xi \in \left(\frac{1}{5}, \frac{1}{4}\right]$ $\quad$ PROB $= \int_{\frac{1}{5}}^{\frac{1}{4}} 1\, ds = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$

$(\xi_1, \xi_2) = (0,1)$ WHEN $\xi \in \left[\frac{2}{3}, \frac{3}{4}\right)$ $\quad$ PROB $= \frac{1}{12}$

$(\xi_1, \xi_2) = (1,1)$ WHEN $\xi \in \left(\frac{1}{4}, \frac{2}{3}\right)$ $\quad$ PROB $= \frac{5}{12}$

JOINT DISTRIBUTION PROBABILITIES ARE

$P(0,0) = \frac{9}{20}$ $\quad P(1,0) = \frac{1}{20}$ $\quad P(0,1) = \frac{1}{12}$ $\quad P(1,1) = \frac{5}{12}$

3.3 c

$\mathrm{COV}(\xi_1, \xi_2)$ WE USE JOINT PROBABILITIES TO CALCULATE IT

$$\mathrm{COV}(\xi_1, \xi_2) = \left(0-\frac{7}{15}\right)\left(0-\frac{1}{2}\right)\cdot\frac{9}{20} + \left(1-\frac{7}{15}\right)\left(0-\frac{1}{2}\right)\cdot\frac{1}{20} +$$

$$+\left(0-\frac{7}{15}\right)\left(1-\frac{1}{2}\right)\frac{1}{12} + \left(1-\frac{7}{15}\right)\left(1-\frac{1}{2}\right)\cdot\frac{5}{12} =$$

$$= \frac{7\cdot 9}{30\cdot 20} - \frac{8}{30\cdot 20} - \frac{7}{30\cdot 12} + \frac{8\cdot 5}{30\cdot 12} =$$

$$= \frac{7\cdot 9\cdot 3 - 8\cdot 3 - 7\cdot 5 + 8\cdot 5\cdot 5}{30\cdot 20\cdot 3} = \frac{330}{1800} = \frac{11}{60}$$

$$\mathrm{VAR}(\xi_1) = \frac{7}{15}\left(1-\frac{7}{15}\right) = \frac{56}{225}$$ ← WE USE THE FORMULA OF THE VARIANCE OF A BERNOULLI.

$$\mathrm{VAR}(\xi_2) = \frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{1}{4}$$

$$\mathrm{CORR}(\xi_1, \xi_2) = \frac{11/60}{\sqrt{\frac{56}{225}\cdot\frac{1}{4}}}$$

SINCE CORR $\neq 0$ WE ARE <u>SURE</u> THAT ξ_1 AND ξ_2 CAN <u>NOT</u> BE INDEPENDENT

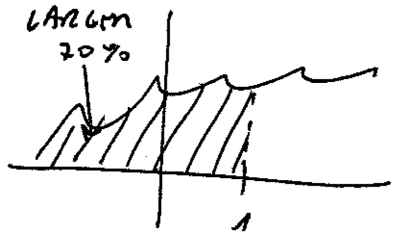
□

3.4 X RANDOM VARIABLE WITH CONT. DISTRIB.

WE KNOW THAT $P(X<1)>0,7$

WHAT CAN WE SAY ABOUT η ?

$$\int_{-\infty}^{1} f(x)\,dx > 0,7$$



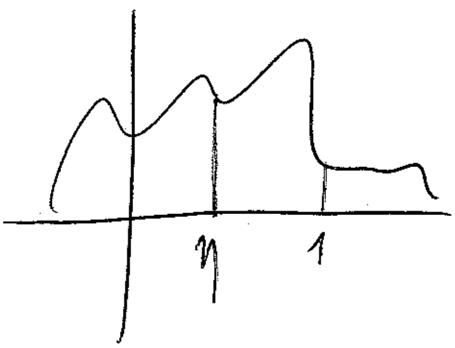
WHAT CAN WE SAY ABOUT $\eta : \int_{-\infty}^{\eta} f(x)\,dx = \frac{1}{2}$?

$$0,7 < \int_{-\infty}^{\eta} f(x)\,dx + \int_{\eta}^{1} f(x)\,dx = \frac{1}{2} + \int_{\eta}^{1} f(x)\,dx$$

$$\Downarrow$$

$$\int_{\eta}^{1} f(x)\,dx > 0,2$$

SINCE $f > 0$ AND THE INTEGRAL IS POSITIVE



$$\Downarrow$$

$$\eta < 1$$

$\square$

3.5

X R.V. CONT. η MEDIAN

$$P(X < \eta^3) = 0.5$$

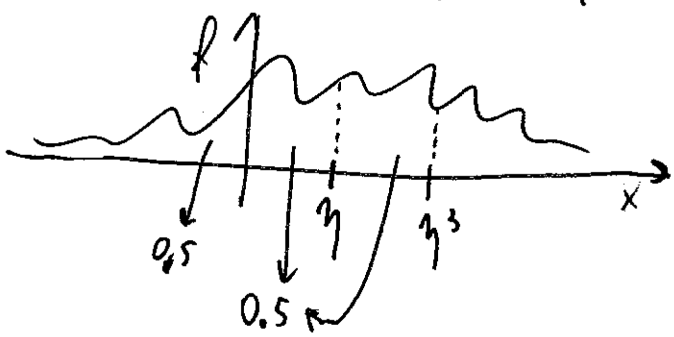
IS IT POSSIBLE ?

$P(X<\eta)=0.5 \qquad P(X<\eta^3)=0.5$

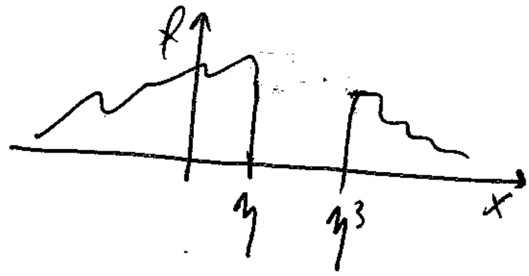
IF $\eta = \eta^3$ IT IS POSSIBLE. THIS MEANS $\eta=1$ OR $\eta=-1$ OR $\eta=0$

OTHERWISE, IF $\eta \neq \eta^3$, IT IS POSSIBLE ONLY IF BETWEEN η AND η^3 (OR BETWEEN η^3 AND η, DEPENDING ON WHICH IS LARGER) THE DENSITY IS 0 AND THEREFORE ANY VALUE BETWEEN η AND η^3 IS A MEDIAN.

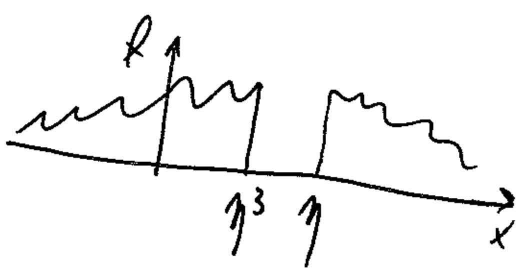
FOR EXAMPLE, IF $\eta > 1$ $(\Rightarrow \eta^3 > \eta)$



$\Rightarrow$



OR, IF $0 < \eta < 1$ $(\Rightarrow \eta^3 < \eta)$



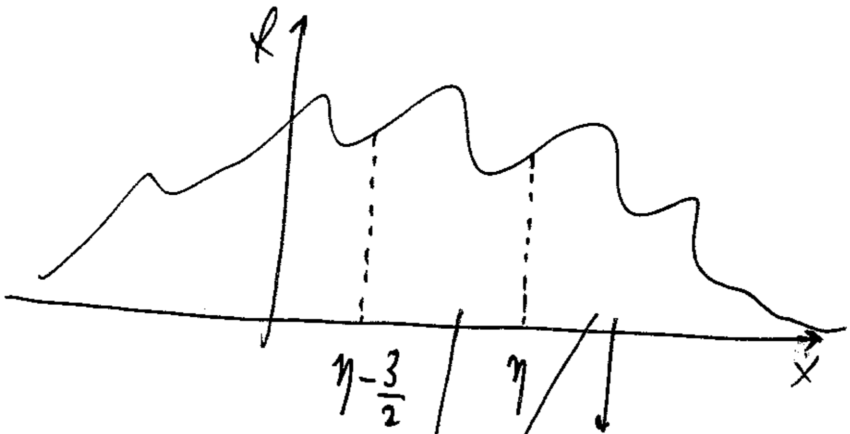
$\square$

3.6)

HW: X cont. r.v. with median η

is it possible that $P(2X+3>2\eta)=0.5$?

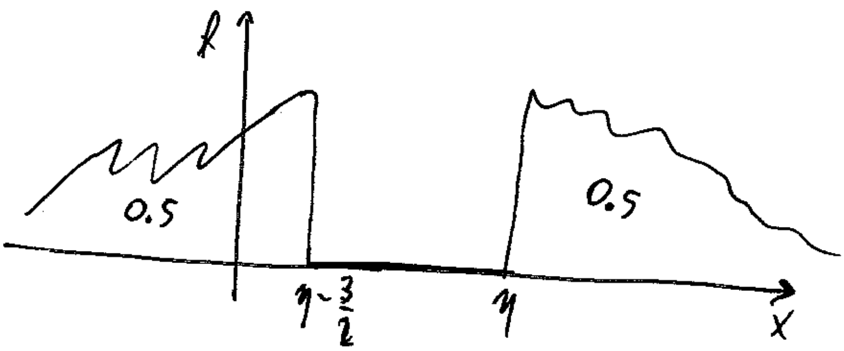
$$P(2X+3>2\eta)=0.5 \Rightarrow P\left(X>\eta-\tfrac{3}{2}\right)=0.5$$



This area is 0.5 due to definition of η $P(X>\eta)=0.5$

This area is 0.5 due to $P\left(X>\eta-\frac{3}{2}\right)=0.5$

Therefore the area between $\eta-\frac{3}{2}$ and η must be equal to 0.



Therefore any number between $\eta-\frac{3}{2}$ and η is a median!

$\Downarrow$

median is not unique.

$\square$

3.7 X, Y CONT. R.V.

$$\forall r \quad P(X < r) < P(Y < r)$$

WHAT CAN WE SAY ABOUT THE MEDIANS η_x η_y?

$$\int_{-\infty}^{r} f_x(s)\,ds < \int_{-\infty}^{r} f_y(s)\,ds \quad \forall r$$

WE TAKE $r = \eta_y$

$$\int_{-\infty}^{\eta_y} f_x(s)\,ds < \int_{-\infty}^{\eta_y} f_y(s)\,ds = \frac{1}{2}$$

$$\int_{-\infty}^{\eta_y} f_x(s)\,ds < \frac{1}{2} = \int_{-\infty}^{\eta_x} f_x(s)\,ds$$

f_x, WHICH IS A POSITIVE FUNCTION, REACHES η_y <u>BEFORE</u> η_x $\implies$ $\eta_y < \eta_x$

$\square$

3.8 FIND THE MEDIAN OF

HW: $$f(x) = \begin{cases} \frac{e^{-\frac{x}{\theta}}}{\theta} & x \geq 0 \\ 0 & x < 0 \end{cases}$$ θ IS A POSITIVE CONSTANT

$$\frac{1}{2} = \int_0^{\eta} \frac{e^{-\frac{x}{\theta}}}{\theta}\, dx =$$ $-\frac{x}{\theta} = s \quad ds = -\frac{dx}{\theta}$

$$= \int_0^{-\frac{\eta}{\theta}} \frac{e^s}{\theta} (-\theta)\, ds = -\int_0^{-\frac{\eta}{\theta}} e^s\, ds = -\left[e^s\right]_0^{-\frac{\eta}{\theta}} = -e^{-\frac{\eta}{\theta}} + 1$$

$$1 - e^{-\frac{\eta}{\theta}} = \frac{1}{2} \qquad e^{-\frac{\eta}{\theta}} = \frac{1}{2} \qquad -\frac{\eta}{\theta} = \ln\frac{1}{2}$$

$$\eta = -\theta \ln\frac{1}{2} = \theta \ln 2$$

D

NOTE: IF WE CALL $\lambda = \frac{1}{\theta}$ THIS IS AN EXPONENTIAL

3.9 WHAT IS THE PROBABILITY THAT A RANDOM VARIABLE EXCEEDS

A) THE MEDIAN OF ITS NORMAL DISTRIBUTION
B) THE EXPECTED VALUE OF ITS NORMAL DISTRIBUTION
C) THE MEDIAN OF ITS DISTRIBUTION
D) THE EXPECTED VALUE OF ITS DISTRIBUTION

GIVEN THE <u>DEFINITION</u> OF MEDIAN $P(X > \eta) = 50\%$.
SO ANSWER TO A) AND C) IS CLEARLY 50%

SINCE FOR $N(\mu, \sigma^2)$ η = EXPECTED VALUE, ANSWER TO B) IS ALSO 50%

FOR D), UNLESS WE KNOW THAT THE DISTRIBUTION IS SYMMETRIC, WE DO <u>NOT</u> HAVE ANY INFORMATION ON PROBABILITIES ON THE RIGHT OF THE EXPECTED VALUE

IT CAN BE VERY LARGE, AS FOR

$$\xi = \begin{cases} -1000 & 1\% \\ 0 & 99\% \end{cases}$$

$$E(\xi) = -1000 \cdot \frac{1}{100} + 0 \cdot \frac{99}{100} = -10 \qquad P(\xi > -10) = 99\%$$

OR IT CAN BE VERY SMALL, AS FOR

$$\xi = \begin{cases} 0 & 99\% \\ 1000 & 1\% \end{cases}$$

$$E(\xi) = 1000 \cdot \frac{1}{100} + 0 \cdot \frac{99}{100} = 10 \qquad P(\xi > 10) = 1\%.$$

<u>THERE IS NO GENERAL RULE FOR NON SYMMETRIC DISTRIBUTIONS</u>

D

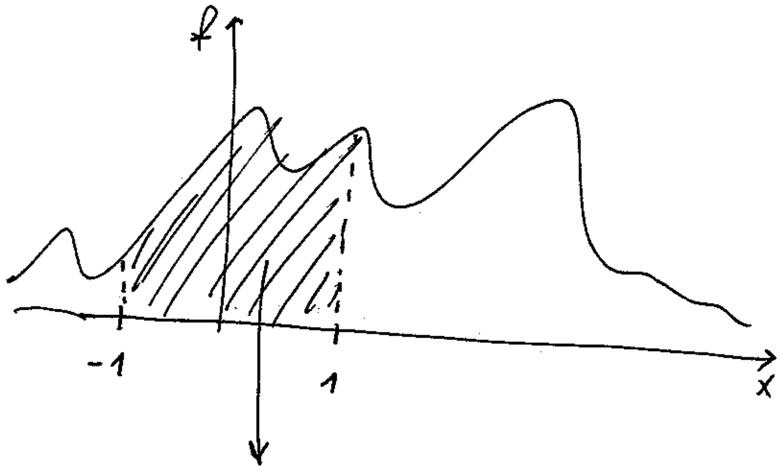
3.10

ξ IS CONTINOUS R.V.

$P(\xi^2 < 1) = 0{,}7$

WHAT CAN YOU SAY ABOUT η MEDIAN ?

$$0{,}7 = P(\xi^2 < 1) = P\{-1 < \xi < 1\} = \int_{-1}^{1} f(s)\,ds$$



70%

THIS MEANS THAT $P(\xi < -1)$ CANNOT EXCEED 30% AND $P(\xi > 1)$ CANNOT EXCEED 30%

$\Downarrow$

η MUST STAY BETWEEN -1 AND 1

IN FACT, IF η WERE < -1, IT WOULD HAVE $P(\xi < \eta)$ LESS THAN 30% AND THIS IS NOT POSSIBLE. THE SAME FOR $\eta > 1$

IF WE WANT AN ANALYTICAL DEMONSTRATION :

$$\int_{-\infty}^{\eta} f\,ds = \frac{1}{2} \Rightarrow \int_{-\infty}^{-1} f\,ds + \int_{-1}^{\eta} f\,ds = \frac{1}{2} \Rightarrow \int_{-1}^{\eta} f\,ds = \frac{1}{2} - \int_{-\infty}^{-1} f\,ds$$

$$\int_{-1}^{1} f\,ds = 0{,}7 \Rightarrow \int_{-1}^{\eta} f\,ds + \int_{\eta}^{1} f\,ds = 0{,}7 \Rightarrow \frac{1}{2} - \int_{-\infty}^{-1} f\,ds + \int_{\eta}^{1} f\,ds = 0{,}7 \Rightarrow \int_{\eta}^{1} f\,ds = 0{,}2 + \int_{-\infty}^{-1} f\,ds > 0{,}2$$

$\Downarrow$

$\boxed{\eta < 1}$ $\square$

3.11 $\xi \sim R(0,1)$ $\quad \varepsilon = -\xi + \xi^2$

FIND $\mathrm{CORR}(\xi, \varepsilon)$

$E(\xi) = \frac{1}{2} \qquad \mathrm{VAR}(\xi) = \frac{1}{12}$

$E(\varepsilon) = \int_0^1 (-s+s^2)\,1\,ds = \left[-\frac{s^2}{2}\right]_0^1 + \left[\frac{s^3}{3}\right]_0^1 = -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}$

$\mathrm{COV}(\xi;\varepsilon) = \int_0^1 \left(s-\frac{1}{2}\right)\left(-s+s^2+\frac{1}{6}\right)\cdot \underset{f(s)}{1}\,ds$

NOTE THAT $f(s)$ IS THE DENSITY OF ξ WHICH IS THE SAME AS ε, SINCE FUNCTIONS OF R.V. DO NOT CHANGE DENSITY

$\mathrm{COV}(\xi;\varepsilon) = \int_0^1 -s^2 + s^3 + \frac{s}{6} + \frac{s}{2} - \frac{s^2}{2} - \frac{1}{12}\,ds = \int_0^1 s^3 - \frac{3}{2}s^2 + \frac{2}{3}s - \frac{1}{12}\,ds =$

$= \left[\frac{s^4}{4} - \frac{3}{6}s^3 + \frac{2}{6}s^2 - \frac{1}{12}s\right]_0^1 = \frac{1}{4} - \frac{3}{6} + \frac{2}{6} - \frac{1}{12} = \frac{3-6+4-1}{12} = 0$

$\mathrm{CORR} = 0$

3.A 12 ξ UNIFORMLY DISTRIBUTED ON $[0,1]$

HW: FIND $\mathrm{CORR}(\xi, e^{\xi})$

SUGGESTION: USE INTEGRATION BY PARTS

$$f_\xi(x) = 1 \qquad f_{e^\xi}(x) = 1$$

$$E(\xi) = \int_0^1 s \cdot 1\, ds = \frac{1}{2} \qquad E(e^\xi) = \int_0^1 e^s \cdot 1\, ds =$$

$$= e - 1$$

$$\mathrm{COV}(\xi, e^\xi) = \int_0^1 \left(s - \frac{1}{2}\right) \cdot \left(e^s - (e-1)\right) \cdot 1\, ds =$$

BY PARTS

$$= \int_0^1 s\, e^s\, ds - \frac{1}{2}\int_0^1 e^s\, ds - \int_0^1 (e-1)s\, ds + \int_0^1 \frac{e-1}{2}\, ds =$$

$$= \left[s\, e^s\right]_0^1 - \int_0^1 e^s\, ds - \left[\frac{e^s}{2}\right]_0^1 - (e-1)\left[\frac{s^2}{2}\right]_0^1 + \frac{e-1}{2} =$$

$$= \cancel{e} - \cancel{e} + 1 - \frac{e}{2} + \frac{1}{2} - \cancel{\frac{e}{2}} + \cancel{\frac{1}{2}} + \cancel{\frac{e}{2}} - \cancel{\frac{1}{2}} =$$

$$= \frac{3}{2} - \frac{e}{2}$$

$$\mathrm{VAR}(\xi) = \int_0^1 \left(s - \frac{1}{2}\right)^2 \cdot 1\, ds = \int_0^1 s^2 - s + \frac{1}{4}\, ds = \left[\frac{s^3}{3} - \frac{s^2}{2} + \frac{s}{4}\right]_0^1 =$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

$\boxed{3.12B}$

$$VAR(e^{q}) = \int_0^1 (e^s - e + 1)^2 \cdot 1\, ds = \int_0^1 e^{2s} + e^2 + 1 - 2ee^s - 2e + 2e^s\, ds =$$

$$= \left[\frac{e^{2s}}{2} + e^2 s + s - 2e e^s - 2es + 2e^s\right]_0^1 =$$

$$= \frac{e^2}{2} + e^2 + 1 - 2e^2 - \cancel{2e} \cancel{+ 2e} - \frac{1}{2} + 2e - 2 =$$

$$= -\frac{e^2}{2} + 2e - \frac{3}{2} \approx 0{,}2479$$

$$CORR = \frac{\frac{3}{\cancel{2}} - \frac{e}{\cancel{2}}}{\sqrt{\frac{1}{\cancel{4}}\frac{1}{3}\left(-\frac{e^2}{2} + 2e - \frac{3}{2}\right)}} = \frac{3 - e}{\sqrt{\frac{4e - e^2 - 3}{6}}} \approx \frac{0{,}2817}{0{,}2840} \approx 0{,}9919$$

$\square$

3.13 $\xi \sim N(0; \sigma_1^2)$ $\varepsilon \sim N(0; \sigma_2^2)$

INDEPENDENT

FIND $\mathrm{CORR}(\xi, a\xi + b\varepsilon)$ WITH $a \in \mathbb{R}$ $b \in \mathbb{R}$

$$E(\xi) = 0 \qquad E(a\xi + b\varepsilon) = aE(\xi) + bE(\varepsilon) = 0$$

SINCE THE JOINT DENSITY OF ξ AND $a\xi+b\varepsilon$ IS VERY COMPLICATED, WE USE THE ORIGINAL DEFINITION OF COVARIANCE AS $\mathrm{COV}(\alpha, \beta) = E\big((\alpha - E(\alpha)) \cdot (\beta - E(\beta))\big)$

$$\mathrm{COV}(\xi, a\xi + b\varepsilon) = E\Big(\big(\xi - E(\xi)\big) \cdot \big(a\xi + b\varepsilon - E(a\xi + b\varepsilon)\big)\Big) =$$

$$= E\big(\xi \cdot (a\xi + b\varepsilon)\big) = E(\xi^2 a) + E(b\xi\varepsilon) = aE(\xi^2) + bE(\varepsilon\xi) =$$

$$= \underbrace{a\,\mathrm{VAR}(\xi)}_{\text{SINCE } E(\xi)=0} + \underbrace{b\,E(\varepsilon) \cdot E(\xi)}_{\text{SINCE THEY ARE INDEPENDENT}} = a\sigma_1^2 + b\,0 \cdot 0 = a\sigma_1^2$$

$$\mathrm{VAR}(a\xi + b\varepsilon) = E\Big(\big(a\xi + b\varepsilon - E(a\xi + b\varepsilon)\big)^2\Big) = E\big((a\xi + b\varepsilon)^2\big) =$$

$$= a^2 E(\xi^2) + b^2 E(\varepsilon^2) + 2ab\,E(\xi \cdot \varepsilon) = a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab \cdot 0 \cdot 0$$

$$\mathrm{CORR} = \frac{a\sigma_1^2}{\sqrt{a^2\sigma_1^2 + b^2\sigma_2^2} \cdot \sqrt{\sigma_1^2}} = \frac{a\sigma_1}{\sqrt{a^2\sigma_1^2 + b^2\sigma_2^2}} = \frac{1}{\sqrt{1 + \left(\frac{b}{a}\frac{\sigma_2}{\sigma_1}\right)^2}}$$

$\square$