12.0 A)

Y_t IS A TIME SERIES OF RANDOM VARIABLES

WEAKLY STATIONARY: - EXP. VALUE DOES NOT DEPEND ON t

- VARIANCE DOES NOT DEPEND ON t
- $\mathrm{COV}(Y_t, Y_{t+s})$ DOES NOT DEPEND ON t BUT CAN DEPEND ON s

WE CALL $\mathrm{COV}(Y_t, Y_{t+s}) = \gamma_s$ (WHICH DEPENDS ONLY ON s)

$\rho_s = \frac{\gamma_s}{\gamma_0}$ IS s-th ORDER AUTOCORRELATION

$\gamma_0 = \mathrm{COV}(Y_t, Y_t) = \mathrm{VAR}(Y_t) \qquad \rho_0 = 1$

WHEN WE HAVE VALUES OF Y_t

$\hat{\gamma}_0 = \frac{1}{n} \sum_{t=1}^{n} (y_t - \bar{y})^2$ IS THE ESTIMATE OF VARIANCE

$\hat{\gamma}_s = \frac{1}{n-s} \sum_{t=1}^{n-s} (y_t - \bar{y}) \cdot (y_{t+s} - \bar{y})$ ARE ESTIMATES FOR COVARIANCE

$\hat{\rho}_s = \frac{\hat{\gamma}_s}{\hat{\gamma}_0}$ PLOTTING $\hat{\rho}_s$ WE OBTAIN THE CORRELOGRAM

EXERCISE 12.1

$\Delta Y_t = Y_t - Y_{t-1}$ IS FIRST DIFFERENCE OF Y_t

Y_t IS $I(1)$ WHEN ΔY_t IS WEAKLY STATIONARY

Y_t IS $I(2)$ WHEN $\Delta(\Delta Y_t) = Y_t - 2Y_{t-1} + Y_{t-2}$ IS WEAKLY STAT.

EXERCISE 12.4

12.0 B)

ε_t IS AR(1) FIRST ORDER AUTOREGRESSIVE

WHEN $\varepsilon_t = \rho\,\varepsilon_{t-1} + \theta_t$

WHERE
- ε_0 DOES NOT DEPEND ON θ_t $\forall t$
- θ_t ARE INDEPENDENT AND $N(0, \sigma_0^2)$

EXERCISE 12.3

NOTE THAT $\varepsilon_1 = \rho\,\varepsilon_0 + \theta_1$ $\quad E(\varepsilon_1) = \rho\,E(\varepsilon_0) + 0$

$$\varepsilon_2 = \rho\,\varepsilon_1 + \theta_2 = \rho^2 \varepsilon_0 + \rho\,\theta_1 + \theta_2 \qquad E(\varepsilon_2) = \rho^2 E(\varepsilon_0) + 0 + 0$$

IN GENERAL $\varepsilon_t = \rho^t \varepsilon_0 + \rho^{t-1}\theta_1 + \dots + \rho^1 \theta_{t-1} + \theta_t$

$$E(\varepsilon_t) = \rho^t E(\varepsilon_0)$$

IF $E(\varepsilon_0) = 0 \Rightarrow E(\varepsilon_t) = 0$ $\forall t$ AND

$$VAR(\varepsilon_t) = VAR(\rho\,\varepsilon_{t-1} + \theta_t) = \rho^2 VAR(\varepsilon_{t-1}) + \sigma_0^2 + $$
$$+ 2\,COV(\rho\,\varepsilon_{t-1}, \theta_t) = \rho^2 VAR(\varepsilon_{t-1}) + \sigma_0^2$$

$COV(\rho\,\varepsilon_{t-1}, \theta_t) = 0$ BECAUSE $\varepsilon_{t-1} = \rho^{t-1}\varepsilon_0 + \rho^{t-2}\theta_1 + \dots + \theta_{t-1}$

AND ε_0 IS INDEPENDENT WITH θ_t AND θ_t ARE INDEPENDENT AMONG THEMSELVES

IF ε_t IS HOMOSKEDASTIC, $VAR(\varepsilon_t) = \sigma^2$ $\forall t$

$$\sigma^2 = \rho^2 \sigma^2 + \sigma_0^2 \iff \sigma^2 = \frac{\sigma_0^2}{1-\rho^2}$$

CONDITION FOR HOMOSKEDASTICITY

$$COV(\varepsilon_t, \varepsilon_{t-1}) = E\big((\varepsilon_t - 0)\cdot(\varepsilon_{t-1} - 0)\big) = E(\varepsilon_t \cdot \varepsilon_{t-1}) = E\big((\rho\,\varepsilon_{t-1} + \theta_t)\cdot \varepsilon_{t-1}\big) =$$
$$= \rho\,E(\varepsilon_{t-1}^2) + E(\theta_t \cdot \varepsilon_{t-1}) = \rho\,\sigma^2 + 0 \cdot 0 = \rho\,\sigma^2$$

12.0 c)

$$\gamma_1 = \text{cov}(\varepsilon_t, \varepsilon_{t-1}) = \rho\,\sigma^2$$

$$\rho_1 = \frac{\gamma_1}{\text{VAR}(\varepsilon_t)} = \rho$$

WE CAN PROVE THAT: IN GENERAL $\gamma_s = \rho^s \sigma^2 \qquad \rho_s = \rho^s$

<u>THEREFORE</u> AR(1) WITH $E(\varepsilon_t) = 0$ AND $\text{VAR}(\varepsilon_t) = \sigma^2 \Rightarrow$ WEAKLY STATIONARY

<u>IF</u> $\rho = 0$ $\quad \varepsilon_t$ IS A <u>WHITE NOISE</u> $\quad \varepsilon_t = \vartheta_t$

<u>IF</u> $\rho = 1$ $\quad \varepsilon_t$ IS A <u>RANDOM WALK</u> $\quad \varepsilon_t = \varepsilon_{t-1} + \vartheta_t$
(IT IS NOT HOMOSKEDASTIC)

<u>IF</u> $\rho = 1$ $\quad \varepsilon_0 = 0$ AND $\vartheta_t \sim N(0, \sigma_\vartheta^2)$ $\quad \varepsilon_t$ IS A <u>BROWNIAN MOTION</u>

EXERCISE 12.2, 12.7, 12.9 HW

HOW TO CHECK THAT AN AR(1) HAS $\rho = 1$?

$$\begin{cases} H_0: \rho = 1 \\ H_1: \rho \neq 1 \end{cases}$$

<u>UNIT ROOT TEST</u>

$$\underbrace{\varepsilon_t - \varepsilon_{t-1}}_{Y_t} = \rho\,\varepsilon_{t-1} + \vartheta_t - \varepsilon_{t-1} = \underbrace{(\rho - 1)\,\varepsilon_{t-1}}_{\delta X_t} + \underbrace{\vartheta_t}_{\vartheta_t}$$

$$Y_t = \delta X_t + \vartheta_t$$

SIMPLE LINEAR REGRESSION. ϑ_t ARE THE ERRORS. AND THEY ARE INDEP.

$$\begin{cases} H_0: \delta = 0 \\ H_1: \delta \neq 0 \end{cases}$$

$$t = \frac{\sqrt{n-2}\; SS_{xy}}{\sqrt{SS_{xx} SS_{yy} - SS_{xy}^2}}$$

t - STUDENT WITH $n-2$ D.F.

WHEN H_0 IS NOT REJECTED, AR(1) HAS $\rho = 1$

EXERCISE 12.6

12.0 D

LAG OPERATOR $L^s(\varepsilon_t) = \varepsilon_{t-s}$

THEREFORE FOR EXAMPLE

$$(3L^2 - 2L + 1)\,\varepsilon_t = 0$$

$$\Updownarrow$$

$$3\varepsilon_{t-2} - 2\varepsilon_{t-1} + \varepsilon_t = 0$$

$$\Updownarrow$$

$$\varepsilon_t = 2\varepsilon_{t-1} - 3\varepsilon_{t-2}$$

DO <u>NOT</u> CONFUSE L WITH Δ

$$\Delta \varepsilon_t = \varepsilon_t - \varepsilon_{t-1}$$

$$\Delta^2 \varepsilon_t = \Delta(\Delta \varepsilon_t) = \Delta(\varepsilon_t - \varepsilon_{t-1}) = \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}$$

EXERCISE 12.5 12.8

12.1 GROSS DOMESTIC PRODUCT FOR U.S.

1952	92,8
1953	94,2
1954	97,4
1955	106,6
1956	109,9
1957	115,4
1958	121,3
1959	128,4
1960	131,0

SUPPOSE IT IS WEAKLY STATIONARY

EVALUATE COEFFICIENTS OF AUTO CORRELATION

PLOT CORRELOGRAM

$\gamma_0 = VAR(Y_t)$ SINCE IT IS WEAKLY STATIONARY, $VAR(Y_t)$ = CONSTANT AND WE ESTIMATE $VAR(Y_t)$ WITH

$$\hat{\gamma}_0 = \frac{1}{n}\sum_{t=1952}^{1960}\left(Y_t - \bar{Y}_9\right)^2 = \frac{1}{9}\sum_{t=1952}^{1960}(y_t - 110,8)^2 = 182,58$$

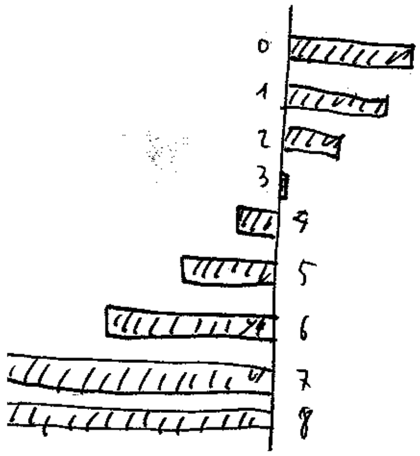
$$\hat{\gamma}_1 = \frac{1}{8}\sum_{t=1952}^{1959}(Y_t - 110,8)(Y_{t+1} - 110,8) = 145,51$$

$$\hat{\gamma}_2 = \frac{1}{7}\sum_{t=1952}^{1957}(Y_t - 110,8)(Y_{t+2} - 110,8) = 83,70$$

$\hat{\gamma}_3 = 10,24$ $\hat{\gamma}_4 = -58,61$ $\hat{\gamma}_5 = -144,32$ $\hat{\gamma}_6 = -250,26$ $\hat{\gamma}_7 = -325,16$

$\hat{\gamma}_8 = -361,55$ $\rightarrow$ WE HAVE TOO FEW DATA TO HAVE A RELIABLE ESTIMATE AFTER $\hat{\gamma}_3$

$$\hat{\rho}_s = (1;\ 0,80;\ 0,46;\ 0,05;\ -0,32;\ -0,79;\ -1,37;\ -1,78;\ -1,98)$$



$\square$

11.2.

e_t ARE R.V. $\sim N(0,1)$

$\varepsilon_0 \sim N(0,\sigma^2)$ INDEP. OF e_t

e_t ARE INDEPENDENT

FIND VALUES FOR σ^2 SUCH THAT ε_t IS W. STATIONARY WHERE $\varepsilon_{t+1} = 0,3\,\varepsilon_t + e_{t+1}$.

FIND COEFFICIENTS OF AUTOCORRELATION

WE MUST PROVE $E(\varepsilon_t)$ CONST, $VAR(\varepsilon_t)$ CONST, $COV(\varepsilon_t, \varepsilon_{t+s})$ DOES NOT DEPEND ON t.

SINCE $\varepsilon_{t+1} = 0,3\,\varepsilon_t + e_{t+1}$, AND ε_0 IND OF e_t, AND e_t INDEP. AND $E(e_t) = 0$ AND $VAR(e_t) = $ CONS

$$\Downarrow$$

ε_t IS $AR(1)$

SINCE $E(\varepsilon_0) = 0 \Rightarrow E(\varepsilon_t) = 0 \quad \forall t$

TO HAVE HOMOSKEDASTICITY, $VAR(\varepsilon_0) = \dfrac{VAR(e_t)}{1-0,3^2}$

$$\sigma^2 = \frac{1}{1-0,09} = \frac{1}{0,91} \Rightarrow \varepsilon_t \text{ IS WEAKLY STATIONARY}$$

MOREOVER, $\gamma_s = \dfrac{0,3^s}{0,91}$ -- $\gamma_0 = 0,91$.

$$\rho_s = 0,3^s = (1; 0,3; 0,09; 0,027 \ldots)$$

(12.3) PROVE THAT Y_t IS $I(1)$ WITH

$$Y_{t+1} = Y_t + \alpha + \varepsilon_t \qquad \varepsilon_t \sim N(0;\sigma^2) \qquad \varepsilon_t \text{ IND } \varepsilon_q \;\; \forall t \neq q$$

IS IT ALSO $AR(1)$?

$I(1) \Leftrightarrow \Delta Y_t$ IS WEAKLY STATIONARY

$$\Delta Y_t = Y_t - Y_{t-1} = \alpha + \varepsilon_{t-1}$$

$E(\Delta Y_t) = E(\alpha + \varepsilon_{t-1}) = \alpha \quad \forall t$ OK

$VAR(\Delta Y_t) = E((\Delta Y_t - \alpha)^2) = E((\alpha + \varepsilon_{t-1} - \alpha)^2) = E(\varepsilon_{t-1}^2) = \sigma^2 \quad \forall t$

$\gamma_0 = \sigma^2$ OK

$COV(\Delta Y_t, \Delta Y_{t+s}) = E((\Delta Y_t - \alpha)(\Delta Y_{t+s} - \alpha)) =$

$= E(\varepsilon_{t-1} \cdot \varepsilon_{t+s-1}) = 0 \quad \forall s \geq 1$ OK IT DOES NOT DEPEND ON t.

IT IS $I(1)$.

TO BE $AR(1)$ $\quad Y_{t+1} = \rho Y_t + \theta_t \quad \theta_t \sim N(0;\sigma^2)$

HERE $\rho = 1$ AND $\theta_t = \alpha + \varepsilon_t$

$\Downarrow$

$\theta_t \sim N(\alpha;\sigma^2) \Rightarrow$ IT IS NOT $AR(1)$ UNLESS $\alpha = 0$.

$\square$

12.4

$Y_t = \alpha + \beta t + \varepsilon_t \quad \varepsilon_t = N(0; \sigma^2) \quad \varepsilon_t$ IND $\varepsilon_q \; \forall t \neq q$

IS IT $I(1)$?

DRAW CORRELOGRAM OF ΔY_t

$$\Delta Y_t = \alpha + \beta t + \varepsilon_t - \alpha - \beta(t-1) - \varepsilon_{t-1} = \beta + \varepsilon_t - \varepsilon_{t-1}$$

$E(\Delta Y_t) = \beta$ OK

$$VAR(\Delta Y_t) = E\left((\varepsilon_t - \varepsilon_{t-1})^2\right) =$$
$$= E(\varepsilon_t^2) + E(\varepsilon_{t-1}^2) + E(-2\varepsilon_t \cdot \varepsilon_{t-1}) =$$
$$= 2\sigma^2 + 0$$

OK $\quad \gamma_0 = 2\sigma^2$

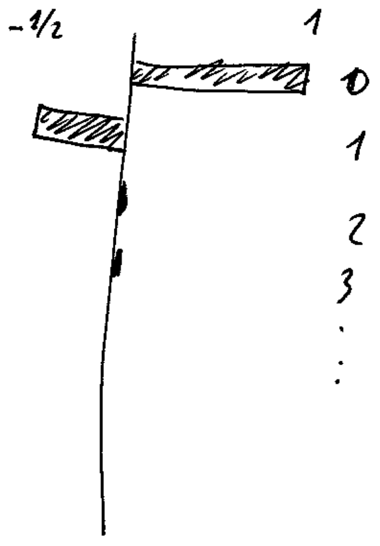
$$COV(\Delta Y_{t+s}; \Delta Y_t) = E\left((\varepsilon_{t+s} - \varepsilon_{t+s-1})(\varepsilon_t - \varepsilon_{t-1})\right) =$$

$$= \varepsilon_{t+s} \cdot \varepsilon_t + \varepsilon_{t+s-1} \cdot \varepsilon_{t-1} - \varepsilon_{t+s-1}\varepsilon_t - \varepsilon_{t+s}\varepsilon_{t-1} =$$

$$= \begin{cases} s=1 & -\sigma^2 \\ s>1 & 0 \end{cases}$$

$\gamma_1 = -\sigma^2 \quad \gamma_s = 0 \; \forall s > 1$ OK!

CORRELOGRAM



D

(12.5) WRITE FINITE DIFFERENCE EQUATION CORRESPONDING TO

$$\left(4L^5 - L^2 + 1\right)\varepsilon_t = e_t$$

$$4\varepsilon_{t-5} - \varepsilon_{t-2} + \varepsilon_t = e_t$$

$$\varepsilon_t = \varepsilon_{t-2} - 4\varepsilon_{t-5} + e_t$$

$$\varepsilon_t - \varepsilon_{t-1} = -\varepsilon_{t-1} + \varepsilon_{t-2} - 4\varepsilon_{t-5} + e_t$$

$$\Delta\varepsilon_t = -\varepsilon_{t-1} + \varepsilon_{t-2} - 4\varepsilon_{t-5} + e_t$$

$$\Delta\varepsilon_t = -\Delta\varepsilon_{t-1} - 4\varepsilon_{t-5} + e_t$$

$\square$

12.6

CONSIDER THIS TIME SERIES

t	1	2	3	4	5	6	7	8
VALUE	7	-3	2	0	2	1	0	-2

SUPPOSE IT IS AR(1)

ESTIMATE ρ AND TEST FOR $\rho = 1$ WITH UNIT ROOT TEST

$$E_t = \rho E_{t-1} + \vartheta_t$$

$$E_t - E_{t-1} = (\rho - 1) E_{t-1} + \vartheta_t$$

$Y_t = E_t - E_{t-1}$ $\quad X_t = E_{t-1}$ $\quad Y_t = (\rho - 1) X_t + \vartheta_t$

t	1	2	3	4	5	6	7	8
Y_t	/	-10	5	-2	2	-1	-1	-2
X_t	/	7	-3	2	0	2	1	0

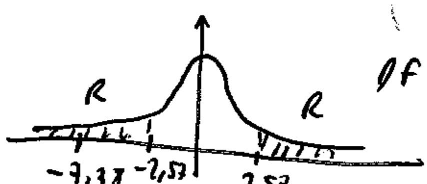
$\bar{X} = \frac{9}{7}$ $\quad \bar{Y} = -\frac{9}{7}$ $\quad SS_{xx} = 7^2 + 3^2 + 2^2 + 0^2 + 2^2 + 1^2 + 0^2 - 7 \cdot \frac{9^2}{7^2} = 55,42$

$SS_{yy} = 127,43$ $\quad SS_{xy} = -10 \cdot 7 + 5 \cdot (-3) - 2 \cdot 2 + 2 \cdot 0 - 1 \cdot 2 - 1 \cdot 1 - 2 \cdot 0 + - 7\left(-\frac{9}{7}\right) \cdot \frac{9}{7} = -80,43$

$\hat{\beta}_1 = -1,45$ $\quad \hat{\rho} = -0,45$

UNIT ROOT TEST: $\begin{cases} H_0: \rho = 1 \\ H_1: \rho \neq 1 \end{cases}$ $\quad t = \frac{\sqrt{n-2}\, SS_{xy}}{\sqrt{SS_{xx} SS_{yy} - SS_{xy}^2}} = \frac{\sqrt{5} \cdot (-80,43)}{\sqrt{55,42 \cdot 127,43 - 80,43^2}} =$

$= \frac{-179,85}{24,36} = -7,38$



DF = 5 $\alpha = 5\%$ REJECTION!

$\rho \neq 1$

12.7

$$\theta_{t+1} = \lambda \theta_t + \varepsilon_{t+1} \qquad \theta_0 = \begin{cases} 2,5 & 50\% \\ -2,5 & 50\% \end{cases}$$

$\varepsilon_t \sim N(0,1)$ INDEP

θ_0 INDEP FROM ε_t

FIND λ SUCH THAT θ_t IS WEAKLY STATIONARY

IN ORDER TO PROVE W.S. WE NEED: $E(\theta_t)$ CONST., $VAR(\theta_t)$ CONST AND $COV(\theta_t, \theta_{t-s})$ INDEP. FROM t.

HOWEVER, IT IS MUCH EASIER TO NOTE THAT θ_t IS AR(1) SINCE:
- $\theta_{t+1} = \lambda \theta_t + \varepsilon_{t+1}$
- ε_t INDEP AND $N(0,1)$
- θ_0 INDEP FROM ε_t

MOREOVER, $E(\theta_0) = 0$ AND SO THE ONLY THING LEFT TO PROVE W.S. IS HOMOSKEDASTICITY.

$$VAR(\theta_0) = \frac{VAR(\varepsilon_t)}{1-\lambda^2} \qquad VAR(\theta_0) = (2,5-0)^2 \cdot 50\% + (-2,5-0)^2 \cdot 50\% =$$

$$= 6,25 \cdot 50\% + 6,25 \cdot 50\% = 6,25$$

$$6,25 = \frac{1}{1-\lambda^2} \qquad 1-\lambda^2 = \frac{1}{6,25} \qquad \lambda^2 = \frac{5,25}{6,25} \qquad \lambda = \pm\sqrt{0,84}$$

WITH $\lambda = \pm\sqrt{0,84}$ WE HAVE AR(1) WITH $E(\theta_0) = 0$ AND HOMOSKEDASTICITY

$\Downarrow$

θ_t WEAKLY STATIONARY

$\square$

12.8

WRITE A FINITE DIFF. EQUATION CORRESPONDING TO $\Delta\theta_t$ WHERE

$$(L^2-2L+1)\theta_t = \varepsilon_t$$

WHAT IS AR(1) MODEL FOR $\Delta\theta_t$?
CAN θ_t BE $I(1)$, $I(2)$ OR $I(k)$?

$$\theta_{t-2}-2\theta_{t-1}+\theta_t = \varepsilon_t$$

EITHER WE REMEMBER THAT $\theta_{t-2}-2\theta_{t-1}+\theta_t$ IS $\Delta(\Delta\theta_t)$ OR WE CAN

$$\theta_t = -\theta_{t-2}+2\theta_{t-1}+\varepsilon_t \implies \theta_t-\theta_{t-1} = -\theta_{t-2}+\theta_{t-1}+\varepsilon_t \implies$$

$$\boxed{\Delta\theta_t = \Delta\theta_{t-1}+\varepsilon_t}$$

THIS IS AR(1) IF $\varepsilon_t \sim N(0,\sigma^2)$ AND $\Delta\theta_1$ DOES NOT DEPEND ON $\varepsilon_t\ \forall t$ AND ε_t ARE INDEPENDENT. UNDER THESE HYPOTHESIS, $\Delta\theta_t$ IS AR(1) AND θ_t IS $I(1)$

CONCERNING $I(2)$, WE TRY TO FIND $\Delta(\Delta\theta_t)$ WHICH IS $\Delta\theta_t-\Delta\theta_{t-1}$

$\Delta(\Delta\theta_t) = \varepsilon_t$. THIS CAN BE AR(1) WITH $\rho=0$ WHEN $\varepsilon_t \sim N(0,\sigma^2)$, $\Delta(\Delta\theta_2)$ DOES NOT DEPEND ON $\varepsilon_t\ \forall t$ AND ε_t ARE INDEPENDENT. UNDER THESE HYPOTHESIS, $\Delta(\Delta\theta_t)$ IS AR(1) AND θ_t IS $I(2)$

CONCERNING $I(3)$, WE TRY TO FIND $\Delta(\Delta(\Delta\theta_t))$, WHICH IS $\Delta(\Delta\theta_t)-\Delta(\Delta\theta_{t-1})$. $\Delta(\Delta(\Delta\theta_t)) = \varepsilon_t-\varepsilon_{t-1}$. IN THIS CASE HOWEVER $(\varepsilon_t-\varepsilon_{t-1})$ CAN NOT BE INDEPENDENT $\forall t \implies$ NO $I(3)$

12.9

$e_t \sim N(0;1)$ INDEP. $\varepsilon_0 \sim N(0;3)$ IND. FROM e_t

FIND VALUES OF λ SUCH THAT ε_t IS WEAKLY STAT.

HW WHERE $\varepsilon_{t+1} = \lambda \varepsilon_t + e_{t+1}$

$e_t \sim N(0;1)$ AND INDEP, ε_0 IS IND FROM e_t, $\varepsilon_{t+1} = \lambda \varepsilon_t + e_{t+1}$

$\Downarrow$

ε_t IS AR(1)

MOREOVER, $E(\varepsilon_0) = 0$. THEREFORE, IN ORDER TO HAVE AUTOMATICALLY HOMOSKEDASTICITY AND $\operatorname{cov}(\varepsilon_t, \varepsilon_{t+s}) = \gamma_s$ WE NEED

$$\operatorname{VAR}(\varepsilon_0) = \frac{\operatorname{VAR}(e_t)}{1-\lambda^2}$$

$$3 = \frac{1}{1-\lambda^2} \qquad 1-\lambda^2 = \frac{1}{3} \qquad -\lambda^2 = -\frac{2}{3} \qquad \lambda^2 = \frac{2}{3}$$

$\lambda = \pm\sqrt{\frac{2}{3}}$ WITH THESE TWO VALUES ε_t IS WEAKLY STAT.

$\square$