13.0

HETEROSKEDASTICITY

IN A REGRESSION MODEL TO PERFORM USEFULNESS TEST WE NEED

E_i INDEPENDENT AND $E_i \sim N(0, \sigma^2)$ WITH SAME VAR.

WHAT HAPPENS IF $VAR(E_i) = \sigma_i^2$?

WE CAN MODIFY THE MODEL IN THIS WAY

$$\frac{Y_i}{\sigma_i} = \frac{\beta_0}{\sigma_i} + \frac{\beta_1}{\sigma_i} X_i + \frac{E_i}{\sigma_i} \qquad E_i^* = \frac{E_i}{\sigma_i} \quad Y_i^* = \frac{Y_i}{\sigma_i} \quad X_i^* = \frac{X_i}{\sigma_i}$$

$$Y_i^* = \frac{\beta_0}{\sigma_i} + \beta_1 X_i^* + E_i^* \quad \text{WITH} \quad VAR(E_i^*) = VAR\left(\frac{E_i}{\sigma_i}\right) = \frac{1}{\sigma_i^2} VAR(E_i) = 1$$

AND WE USE THIS MODEL, UNFORTUNATELY WITH $\frac{\beta_0}{\sigma_i}$ DEPENDENT ON i, AND WE MUST KNOW σ_i IN ADVANCE!

HETEROSKEDASTICITY IS TYPICAL WHEN Y_i ARE OBTAINED AGGREGATING (SUMMING OR TAKING AVERAGE) QUANTITIES ON SAMPLES WITH DIFFERENT NUMBER OF ELEMENTS!

13.1 A WE TAKE ITALIAN FASHION COMPANIES AND CONSIDER THEIR AGGREGATED SALES BY YEAR, WHILE THE NUMB OF COMPANIES IN OUR SAMPLE MAY CHANGE (DUE TO JOINT OR BANKRUPCY)

ITALIAN GDP	1191	1248	1295	1335	1428	1479
FASHION SALE	175	181	192	211	235	255
NUMB. OF COMP.	20	15	21	18	20	25

BUILD A REGRESSION MODEL TO PREDICT SALES USING GDP, ASSUMING THAT THE ERROR VARIANCE OF EACH COMPANY IN THE SAMPLE IS CONSTANT σ (BUT NOT THE ERROR VARIANCE OF THE SUM OF COMPANIES)

FOR EACH COMPANY $Y_{ij} = \beta_0 + \beta_1 X_j + E_{ij}$

SUMMING $Y_j = \sum_{i=1}^{n_j} Y_{ij} = \sum_{i=1}^{n_j} \beta_0 + \sum_{i=1}^{n_j} \beta_1 X_j + \sum_{i=1}^{n_j} E_{ij} =$

$= \beta_0 n_j + \beta_1 n_j X_j + E_j$

WHERE $VAR(E_j) = VAR\left(\sum_{i=1}^{n_j} E_{ij}\right) = n_j VAR(E_{ij}) = n_j \sigma^2$

WE HAVE HETEROSKEDASTICY!

DIVIDING EACH EQUATION BY $\sqrt{n_j}$ $\quad \frac{Y_j}{\sqrt{n_j}\sigma} = \frac{\beta_0 n_j}{\sqrt{n_j}\sigma} + \frac{\beta_1 n_j X_j}{\sqrt{n_j}\sigma} + \frac{E_j}{\sqrt{n_j}\sigma}$

13.1) B

$$Y_j^* = \frac{Y_j}{\sqrt{n_j}\,\sigma} \qquad X_j^* = X_j \frac{\sqrt{n_j}}{\sigma} \qquad w_j = \frac{\sqrt{n_j}}{\sigma}$$

$$\varepsilon_j^* = \frac{\varepsilon_j}{\sqrt{n_j}\,\sigma}$$

MODEL BECOMES $Y_j^* = \beta_0 w_j + \beta_1 X_j^* + \varepsilon_j^*$

$$VAR(\varepsilon_j^*) = \left(\frac{1}{\sqrt{n_j}\,\sigma}\right)^2 VAR(\varepsilon_j) = \frac{n_j\sigma^2}{(\sqrt{n_j}\,\sigma)^2} = \sigma^2 \text{ CONSTANT}$$

WE INTRODUCE HOWEVER ALSO A NEW CONSTANT PARAMETER β_2

$$Y_j^* = \beta_2 + \beta_0 w_j + \beta_1 X_j^* + \varepsilon_j^*$$

NOW WE ESTIMATE PARAMETERS USING LEAST-SQUARES

$$SSE = \sum_{j=1}^{6}\left(Y_j^* - \hat{Y}_j^*\right)^2 = \sum_{j=1}^{6}\left(Y_j^* - \beta_2 - \beta_0 w_j - \beta_1 X_j^*\right)^2$$

$$\begin{cases} \dfrac{\partial SSE}{\partial \beta_2} = -2\sum_j \left(Y_j^* - \beta_2 - \beta_0 w_j - \beta_1 X_j^*\right) = 0 \\ \dfrac{\partial SSE}{\partial \beta_0} = -2\sum_j \left(\quad\right)\cdot w_j = 0 \\ \dfrac{\partial SSE}{\partial \beta_1} = -2\sum_j \left(\quad\right)\cdot X_j^* = 0 \end{cases}$$

$$\begin{cases} 6\beta_2 + \sum_j w_j\,\beta_0 + \sum_j X_j^*\,\beta_1 = \sum_j Y_j^* \\ \sum_j w_j\,\beta_2 + \sum_j w_j^2\,\beta_0 + \sum_j X_j^* w_j\,\beta_1 = \sum_j Y_j^* w_j \\ \sum_j X_j^*\,\beta_2 + \sum_j w_j X_j^*\,\beta_0 + \sum_j X_j^{*2}\,\beta_1 = \sum_j Y_j^* X_j^* \end{cases}$$

$$\begin{cases} 6\beta_2 + 26{,}6\,\beta_0 + 35539{,}4\,\beta_1 = 280{,}8 \\ 26{,}6\,\beta_2 + 119\,\beta_0 + 159300\,\beta_1 = 1248 \\ 35539{,}4\,\beta_2 + 159300\,\beta_0 + 214499460\,\beta_1 = 1676068 \end{cases}$$

$\beta_2 = 87{,}2$

$\beta_0 = -26{,}6$

$\beta_1 = 0{,}013$

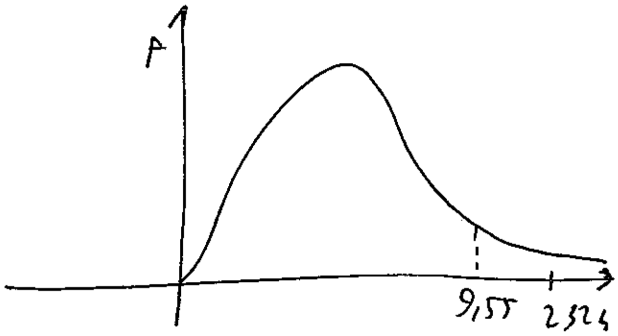
$SSE = 8{,}569$

$SS_{y^*y^*} = 13288{,}1$

$$r^2 = 1 - \frac{SSE}{SS_{y^*y^*}} = 1 - \frac{8{,}569}{13288{,}1} = 0{,}99935$$

13.1 c)

$$F = \frac{\frac{r^2}{2}}{\frac{1-r^2}{6-3}} = 2324$$



$DF = 2,3$

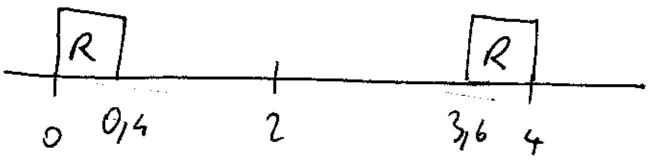
$\alpha = 5\%$

H_0 IS REJECTED $\Rightarrow$ MODEL IS USEFUL.

IN ORDER TO CHECK ALSO FOR INDEPENDENCE OF E^*_j

$$d = \frac{\sum_{j=2}^{6}\left(\hat{E}^*_j - \hat{E}^*_{j-1}\right)^2}{\sum_{j=1}^{6} \hat{E}^{*2}_j} = \frac{10,955}{8,569} = 1,278$$

$\alpha = 5\%$



WE DO NOT REJECT H_0

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ERRORS ARE NOT AUTOCORRELATED

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E^*_j ARE INDEPENDENT

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