# Contents

# Introduction

Turbulence models have become nowadays more and more important in fluid dynamics computation, as the numerical approximation of the pure Navier-Stokes equations becomes unstable for large values of Reynolds number. This thesis deals with the mathematical problem of turbulence in the hydraulics of incompressible fluids. We will consider three related but different problems. First of all, we will analyze analytical and numerical properties of $\varphi$-$\theta$ model, a two-equations Boussinesq's turbulence model which is alternative to the $k$-$\epsilon$ one and has much better stability and positivity properties. We will then analyze Large Eddy Simulation (LES) models in various versions: with or without space filtering terms and with or without eddy viscosity. We will study analytical properties of space filtering (SF) model and then analytical properties of eddy viscosity space filtering (EVSF) model, providing conditions for existence and uniqueness of solution. Finally, we will provide a finite elements numerical algorithm to solve LES models and we will show its performance in the cavity test problem.

The problem of turbulence in hydraulics arises from Navier-Stokes equations which describe the flow of an incompressible fluid. It is still unknown an analytical way of solving these equations except in very simple cases and therefore we must rely on numerical approximation their solution. Consistent algorithm to solve these equations prove to be unstable when viscosity is small enough to let Reynolds number be above $10^3$ unless a very fine space mesh is used. On the other hand, experiments show that the flow is no more laminar at these Reynolds number and highly fluctuating fluid velocities appear. This physical and mathematical effect is called turbulence. It causes a subtraction of energy from the mean motion which causes the high fluctuation of velocity and is then dissipated by viscosity forces. This effect can be roughly represented by an increase in kinematic viscosity called turbulent viscosity.

The Reynolds mathematical approach to the problem of turbulence is to average Navier-Stokes equations over time or over a set of experimental data and, using average properties, reducing Navier-Stokes equations to the formally identical Reynolds equations with a Reynolds stress tensor as an additional term. This tensor is later approximated by Boussinesq using the eddy viscosity concept. The determination of eddy viscosity's value is a problem in itself: it can be taken constant, depending on fluid's velocity or taken as a solution of one or two differential equations. The $k$-$\epsilon$ two differential equations model is the most popular and the one which gives the best numerical results. However it suffers stability and positivity problems, since an analytical proof of its positivity, on which a positive numerical algorithm can be based, is given only in very regular cases. In this work we use the $\varphi$-$\theta$ two equations model proposed by Mohammadi [32], which is based on $k$-$\epsilon$ model changing its variables and approximating viscous terms; this model has much better stability and positivity properties and, starting from an analytical proof of its existence, uniqueness, stability and positivity given by Mohammadi [32], we prove the positivity and stability of a suitable numerical scheme.

Another mathematical approach to the problem of turbulence is to average Navier-Stokes equations on space instead of time using convolutions and Fourier transform instead of very simple ensemble average. We now cannot reduce convective averaged term as a sum of a convective-like term and a stress tensor, as we did in Reynolds equations, but we have to approximate each part of the convective averaged term. Using a Taylor expansion, where expanded terms do not involve highly fluctuating velocities, and a Taylor expansion of the convolution filter, where expanded terms are highly fluctuating, we obtain in this way space filtering (SF) large eddy simulation turbulence model. For the analytical solution of this model we provide an existence and uniqueness problem provided that initial data is regular and small enough. To do this we use Schauder's fixed point theorem and, relying on the fact that initial velocity is small enough in a quite high Sobolev norm, we manage to control the nonlinear space filtering and convective terms with the positive viscous term thus satisfying the main condition of Schauder's theorem. The numerical approximation of SF model using a three-steps time advancement, which splits SF model into two Stokes problems and one Burgers problem which is resolved using an iterative method, proves to be rather inefficient compared to the same method used to solve Navier-Stokes equations: it needs a much smaller time step to converge at low Reynolds numbers, while it does not work anymore at moderate Reynolds number. This is coherent to the smallness condition on initial velocity which appears in existence theorem, which is no more satisfied when Reynolds number is too large, and therefore the positive viscous term cannot control the nonlinear space filtering term anymore.

Due to the poor numerical properties of SF model, it is common to approximate the averaged convective term with a convective-like term and an eddy viscosity term, obtaining eddy viscosity (EV) large eddy simulation model. Another possibility is to add again the space filtering term obtaining therefore the eddy viscosity space filtering (EVSF) large eddy simulation model. Since an eddy viscosity, depending on the norm of the mean velocity gradient, is now present, these models have much better analytical and numerical properties than SF model. Following the proof of existence and uniqueness of analytical solution of EV model given by Ladyzhenskaya [23] we managed to prove existence and uniqueness of analytical solution of EVSF model using a Galerkin approximation method and using eddy viscosity term to control, in an high enough Sobolev norm, the space filtering term. We need to require that eddy viscosity coefficient is large enough compared to the space filtering coefficient. Convective terms are then controlled by another part of eddy viscosity term using Gronwall's lemma. This proof does not need a small initial data and the condition on eddy viscosity and space filtering coefficient is compatible with the usual values taken for them in numerical simulations. The previous algorithm proved to be quite efficient with EV model and it works for every Reynolds number up to $10^6$ with quite small eddy viscosity coefficient. However EVSF model needs an eddy viscosity constant too large and a time step too small to be considered efficient.

In Chapter 1 we present the physical and mathematical description of the problem of turbulence and the derivation of Reynolds equations relying on strong ensemble average properties. Then we introduce Boussinesq's approximation presenting its main criticisms, an original generalization and a small correction, concluding with one and two equations turbulence models. Then we take the other approach to the mathematics of turbulence: we present the properties of convolution and Fourier transform and various examples of filters which are used to obtain Leonard's approximation, Clark et al.'s approximation an Yeo and Bedford's model which constitute SF model and Smagorinsky's model which gives EV and EVSF models.

In Chapter 2 we build, starting from $k$-$\epsilon$ model, $\varphi$-$\theta$ model and approximate its viscous terms to obtain two simpler equations. After presenting Gronwall's lemmae we give a rapid sketch of Mohammadi's proof of existence and uniqueness of $\varphi$-$\theta$ model's analytical solution. Then we discretize spatially $\varphi$-$\theta$ model with finite elements obtaining a positive and stable numerical solution. We switch then to time and space discretization and provide a numerical algorithm showing existence, positivity and stability.

In Chapter 3 we present the existence theorem for SF model. We define the starting set for Schauder's theorem and show some control lemmae to assure boundedness of high norms of solution. After showing compactness of starting set and existence and continuity of Schauder's map, we conclude with a fixed point argument. Then we easily prove uniqueness and stability of solution and the existence, under certain condition on external forces, of periodic and stationary solutions.

In Chapter 4 we start by presenting two a priori estimates of solution using an eddy viscosity coefficient large enough; in order to deal with space filtering term, we have to sum two different estimates to be able to prove the second a priori estimates on high Sobolev norms of velocity. Then we use a Galerkin technique to build a sequence of functions which converges to the solution of EVSF model thanks to the control of viscous terms over space filtering and convective terms. In order to deal with the highly nonlinear terms involved we follow an idea of Minty and Browder to preserve the positive behaviour of viscous term. Then we prove uniqueness, stability and decay of solution under certain conditions on external forces.

In Chapter 5 we present a numerical scheme for Navier-Stokes, SF, EV and EVSF models based on a three-step time splitting scheme proposed by Glowinsky [18] which reduces our problem to two Stokes problems and one Burgers problem. We introduce Taylor-Hood finite elements which satisfy Ladyzhenskaya-Babuška-Brezzi condition and we provide an iterative conjugate gradient method to solve the linear systems derived from Stokes' problems. Then we present an iterative nine-steps version of conjugate gradient proposed by Girault and Raviart [17] to solve the nonlinear Burgers problem. We illustrate our test problem, the two dimensional cavity flow, and the results obtained for low Reynolds numbers compared to results obtained by Ghia et al. for Navier-Stokes with a very fine mesh. SF model proved to be not worth the effort for low Reynolds numbers and does not work, with our numerical scheme, for moderate and large Reynolds number, while Navier-Stokes equations provide good agreement with Ghia at al.'s data. For large Reynolds numbers we do not have comparative data anymore but EV model proves to be quite efficient, while EVSF problem needs a too much large eddy viscosity constant to work.

I would like to thank Prof. Alberto Valli to have dedicated to this work and to my numerical and analytical problems a lot of his time and of his efforts. I would also thank Dr. Rosa Loredana Trotta, Prof. Roger Lewandowski, Prof. Aronne Armanini, Prof. Olivier Pironneau and the whole Department of Mathematics of the University of Trento for their scientific help and my mother and my father for their personal support.

# Glossary

| | |
|---|---|
| $u$ | instantaneous velocity |
| $U$ | mean velocity |
| $u'$ | turbulent (fluctuating) velocity |
| $\tau$ | Reynolds stress tensor |
| $Re$ | Reynolds number |
| $t$ | time |
| $N$ | the dimension of $\mathbb{R}^N$, in our cases $N = 2$ or $N = 3$ |
| $x_i$ | i-th coordinate axis |
| $\nu$ | kinematic viscosity |
| $\nu_t$ | turbulent eddy viscosity |
| $\partial_i$ | partial derivative with respect to $i$-th coordinate |
| $\partial_t$ | partial derivative with respect to time |
| $D$ | mean velocity deformation tensor, $D_{ij} = 1/2[\partial_j U_i + \partial_i U_j]$ |
| $\nabla$ | gradient, $(\partial_1, \ldots, \partial_N)$ |
| $\Delta$ | Laplace operator, $\sum_{j=1}^{N} \partial_j \partial_j$ |
| $d_t$ | total time derivative, $\partial_t + \sum_{j=1}^{N} U_j \partial_j$ |
| $f$ | external forces |
| $P$ | mean pressure |
| $p$ | instantaneous pressure |
| $p'$ | turbulent (fluctuating) pressure |
| $k$ | turbulent kinetic energy (per unit mass) |
| $\epsilon_t$ | turbulent kinetic energy dissipation (per unit mass) |
| $\rho$ | density |
| $g$ | gravity acceleration |
| $\mathbb{R}$ | set of real numbers |
| $\mathbb{N}$ | set of natural numbers (zero excluded) |
| $\mathbb{Q}$ | set of rational numbers |
| $\mathbb{P}^k$ | set of polynomials of degree $k$ |

| | |
|---|---|
| $\lambda$ | spatial filter width |
| i | imaginary unit, $\sqrt{-1}$ |
| $\Omega$ | an open, connected and bounded subset of $\mathbb{R}^N$ with regular boundary |
| $\partial\Omega$ | the boundary of $\Omega$ |
| $\eta^+$ | the positive part of function $\eta$, $\max(0,\eta)$ |
| $\eta^-$ | the negative part of function $\eta$, $\max(0,-\eta)$ |
| $L^p$ | space of functions whose $p$-th power is integrable |
| $L^\infty$ | space of almost always bounded functions |
| $W^{m,p}$ | space of functions in $L^p$ with distributional derivatives up to order $m$ in $L^p$ |
| $W^{m,\infty}$ | space of functions in $L^\infty$ with distributional derivatives up to order $m$ in $L^\infty$ |
| $H^m$ | space of functions in $L^2$ with distributional derivatives up to order $m$ in $L^2$, $W^{m,2}$ |
| $H^m_0$ | space of functions in $H^m$ with zero trace on the boundary |
| $L^2_0$ | space of $L^2$ functions with a vanishing average |
| $A(X;B)$ | space of functions whose $B$-norm is in $A(X)$ |
| $h$ | spatial grid size, the largest diameter of spheres circoscribing each element |
| $\Delta t$ | temporal grid size, $t^{n+1}-t^n$ |
| SF | space filtered LES model (1.44) |
| EV | eddy viscosity LES model, (1.47) with $\lambda=0$ |
| EVSF | eddy viscosity space filtered LES model (1.47) |

# Chapter 1

# Turbulence

In this chapter we are going to review the problem of turbulence in hydraulics, its physical characteristics and the models adopted to describe turbulence effects. We will spend more words on $k$-$\epsilon$ and on Large Eddy Simulation models, since these are the most widely used and their analytical and numerical properties are discussed in the other chapter of this work.

## 1.1 The equations of Navier-Stokes

The equations which describe the motion of an isotropic, incompressible newtonian fluid in a time-independent domain are the well-known Navier-Stokes equations

$$\begin{cases} \partial_t u_i + \sum_{j=1}^{N} u_j \partial_j u_i = -\frac{1}{\rho}\partial_i p + \nu \Delta u_i + f_i & i = 1, \dots, N \\ \sum_{j=1}^{N} \partial_j u_j = 0, \end{cases} \tag{1.1}$$

where $N$ is the dimension of the domain where the fluid is contained, $\partial_t$ is the derivative with respect to time, $u_i$ the velocity component, $\partial_i$ the derivative with respect to $x_i$, $\nu$ the kinematic viscosity, $\Delta$ the Laplacien operator, i.e. $\sum_{i=1}^{N} \partial_i \partial_i$, $\rho$ the density of the fluid, which is assumed to be constant, $p$ its pressure and $f_i$ is the component of the external force. We will call the second term of the first $N$ equations *convective term*, the third term *pressure term* and the fourth term *viscous term*. The last equation of (1.1) is called the *continuity equation*.

These equations must be considered together with boundary and initial conditions in order to obtain a well-posed Cauchy problem. Usually, we give as initial condition the value of velocity in every point of domain at initial time. Boundary conditions vary according to the problem: we can give the value of velocity at every point on the boundary if the domain is bounded, or we can give information on the limit value of velocity as distance from fixed point goes to infinity. In both these cases we have to give a condition on pressure to assure its uniqueness: we can do this by stating that its average pressure on the domain is zero.

## 1.2 The problem of turbulence

If we take Navier-Stokes equations (1.1), multiply the first $N$ equations by $L/V^2$, where $V$ is a *scale velocity* and $L$ is a *scale length*, and we multiply the continuity equation by $L/V$, we obtain

$$\begin{cases} \dfrac{\partial \tilde{u}_i}{\partial \tilde{t}} + \displaystyle\sum_{j=1}^{N} \tilde{u}_j \dfrac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\dfrac{1}{\rho}\dfrac{\partial \tilde{p}}{\partial \tilde{x}_i} + \dfrac{\nu}{LV}\displaystyle\sum_{j=1}^{N} \dfrac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2} + \dfrac{L}{V^2} f_i \qquad i = 1, \dots, N \\ \displaystyle\sum_{j=1}^{N} \dfrac{\partial \tilde{u}_j}{\partial \tilde{x}_j} = 0, \end{cases} \tag{1.2}$$

where $\tilde{u} = u/V$ is adimensional velocity, $\tilde{t} = tV/L$ adimensional time, $\tilde{x} = x/L$ adimensional length, $\tilde{p} = p/V^2$ adimensional pressure and

$$Re = \frac{LV}{\nu} \tag{1.3}$$

is called the Reynolds number. It is immediately clear that, when Reynolds number tends to infinity, Navier-Stokes equations tend to Euler equations, while when Reynolds number tends to zero, or at least when it is small enough, the viscous term becomes preponderant.

Navier-Stokes (1.1) are the equations describing the motion of every isotropic incompressible newtonian fluid since they are derived directly from conservation laws without further assumptions. However, unless in very easy situations like Poueisille's motion or Couette's motion, we do not know the analytical solution of this problem. Existence and uniqueness of a solution of the Cauchy problem for Navier-Stokes in a bounded domain has been proved in the two-dimensional case, and only existence and uniqueness for small Reynolds initial numbers has been proved in the three-dimensional case [20].

Therefore several numerical procedures to solve these equations have been developed. However, they have the same problems as analytical solutions: they are highly unstable as Reynolds number becomes large. We call this effect *turbulence.* In order to have more stable numerical solution, i.e. to study the turbulent motion, the mesh size of the numerical grid need to be smaller, typically $10^{-3}$ times, than the size of the domain [40]. This means $10^{3N}$ grid points where $N$ velocity components and pressure data need to be stored; this is still far beyond the capacity of present-day computers and, in addition, the number of arithmetic operations which would be required is so large that the computing time would also be prohibitive.

Since engineers nevertheless need calculation methods, they resort to empirical and semi-empirical methods. Empirical methods simply correlate experimental results and can therefore be used with confidence only for direct interpolation of these results. Because there was and there is little hope of solving the complete set of equations and because engineers are in any case not interested in the details of the turbulent motion but only in its mean flow, a statistical approach has been at first suggested by Osborne Reynolds [39].

## 1.3 A phenomenological description

Most flows occurring in nature and engineering applications are turbulent. The boundary layer in the earth's atmosphere is turbulent; jet streams in the upper troposphere are turbulent; cumulus clouds are in turbulent motion; the water currents below the surface of the oceans are turbulent; the flow of water in rivers and canals is turbulent; the wakes of ships, cars, submarines and aircraft are in turbulent motion. In fluid dynamics laminar flow is an exception, not the rule: we must have small dimensions and high viscosities to encounter laminar flow. The flow of lubricating oil in a bearing is a typical example.

In flows which are originally laminar, turbulence arises from instabilities at large Reynolds numbers. Laminar pipe flows become turbulent at a Reynolds number based on mean velocity and diameter in the neighborhood of 2000 unless great care is taken to avoid creating small disturbances that might induce transition from laminar to turbulent flow.

On the other hand, turbulence cannot maintain itself but depends on its environment to obtain energy. A common source of energy for turbulent velocity fluctuations is shear in the mean flow; other sources, such as buoyancy, exist too. Turbulent flows are generally shear flows; if turbulence arrives in an environment where there is no shear or other maintenance mechanism, it decays: the Reynolds number decreases and the flow tends to become laminar again. The classic example is turbulence produced by a grid in uniform flow in a wind tunnel.

Mathematically, the details of transition from laminar to turbulent flow are rather poorly understood. Much of the theory of instabilities in laminar flows is linearized theory, valid for very small disturbances; it cannot deal with the large fluctuation levels in turbulent flow. On the other hand, almost all the theory of turbulent flow is asymptotic theory, fairly accurate at very high Reynolds numbers but inaccurate and incomplete for Reynolds numbers at which turbulence cannot maintain itself.

Experiments have shown that transition is commonly initiated by a primary instability mechanism, which in simple cases is two-dimensional. The primary instability produces secondary motions, which are generally three-dimensional and become unstable themselves. A sequence of this nature generates intense localized three-dimensional disturbances which arise at random positions at random times. These spots grow rapidly and merge with each other when they become large and numerous to form a field of developed turbulent flow. In other cases, turbulence originates from an instability that causes vortices which subsequently become unstable: this phenomenon is known as vortices cascade.

### 1.3.1 The features of turbulence

Everybody has some idea about the nature of a turbulent flow, however, it has not a precise mathematical definition and it is very difficult to provide a physical one. All we can do is to list some of the characteristics of turbulent flows.

- One characteristic is the **irregularity**, or randomness. This makes a deterministic approach to turbulence problems impossible; instead, we rely on statistical methods.
- The **diffusivity** of turbulence, which causes rapid mixing and increased rates of momentum, heat and mass transfer, is another important fea-

ture of all turbulent flows. If a flow pattern looks random but does not exhibit spreading of velocity fluctuations through the surrounding fluid, it is surely not turbulent. The diffusivity of turbulence is the single most important feature as far as applications are concerned: it increases heat transfer rates in machinery of all kind, it is the source of the resistance of flow in pipelines and it increases momentum transfer between winds and ocean currents.

- Turbulent flows always occur at **large Reynolds numbers**. Turbulence often originates as an instability of laminar flows if the Reynolds number becomes too large. The instabilities are related to the interaction of viscous terms and nonlinear inertia terms in the equations of motion. This interaction is very complex: the mathematics of nonlinear partial differential equations has not been developed to a point where general solutions can be given. Randomness and nonlinearity combine to make the equations of turbulence nearly intractable; turbulence theory suffers from the absence of sufficiently powerful mathematical methods. This lack of tools makes all theoretical approaches to problems in turbulence trial-and-error affairs.

- Turbulence is **rotational**. Turbulence is characterized by high level of fluctuating vorticity. For this reason, vorticity dynamics play an essential role in the description of turbulent flow.

- Turbulent flows are always **dissipative**. Viscous shear stresses perform deformation work which increases the internal energy of the fluid at the expense of kinetic energy of the turbulence. Turbulence needs a continuous supply of energy to make up for these viscous losses. If no energy is supplied, turbulence decays rapidly. Random motions, such as acoustic noise (random sound waves), have insignificant viscous losses and, therefore, are not turbulent.

- Turbulence is a **continuum** phenomenon, governed by the equations of fluid mechanics. Even the smallest scale occurring in a turbulent flow are ordinarily far larger than any molecular length scale.

- Turbulence is not a feature of fluids but of **fluid flows**. Most the dynamics of turbulence is the same in all fluids, whether they are liquids or gases, if the Reynolds number is large enough; the major characteristics of turbulent flows are not controlled by the molecular properties of the fluid in which the turbulence occurs. Since every flow is different, it follows that every turbulent flow is different, even though all turbulent flows have many characteristics in common.

## 1.4 Reynolds statistical approach

The statistical approach for the study of turbulence starts from the idea that turbulent fluctuations of velocity have a smaller time and length scale than the scale of velocity of practical interest to engineers. Therefore the idea of Reynolds was to average Navier-Stokes equations (1.1) over a time scale long compared with that of the turbulent motion. The resulting equations describe the distribution of mean velocity and pressure in the flow and thus the quantities of prime interest to the engineer. Unfortunately, the process of averaging creates a new problem: now the equations no longer constitute a closed system since they contain unknown terms representing the transport of mean momentum by turbulent motion.

Formally, Reynolds equations (1.7) are obtained from Navier-Stokes by considering the velocity $u$ and the pressure $p$ as a sum of a mean component, in capital, and a turbulent component, indicated with a prime, i.e.

$$u_i = U_i + u'_i \qquad\qquad p = P + p'. \tag{1.4}$$

We have to give a definition of *mean component*. This can be done by setting a Cauchy problem and performing many physical experiments on that problem; in this way we can measure velocity and pressure at every time on every point and obtaining mean components by averaging over the set of experimental data. By difference with instantaneous quantities we can define turbulent quantities too. Therefore the mean of a mean quantity is assumed to be the mean quantity itself and the mean of a turbulent quantity is zero.

However, if experimental data are not available and if we do not want to base our theory on only potentially-existent data, we can revert to the original idea of Reynolds: averaging over a long time scale. In this way the mean quantities are defined as

$$U_i(x_j, \bar{t}) = \frac{1}{T}\int_{\bar{t}}^{\bar{t}+T} u_i(x_j, t)\, dt \qquad\qquad P(x_j, \bar{t}) = \frac{1}{T}\int_{\bar{t}}^{\bar{t}+T} p(x_j, t)\, dt, \tag{1.5}$$

where $T$ has to be large compared with the time scale of the turbulent motion and, if the flow is not stationary, small compared with the time scale of the mean flow. As usual, turbulent quantities are derived by difference using (1.4). However, with this definition the average of a mean quantity is not the mean quantity itself, unless the flow is stationary, and therefore the average of a turbulent quantity is not zero. In any case, since $T$ is small compared with the time scale of the mean flow, we will assume that the average of a turbulent quantity is zero and the average of a mean quantity is the quantity itself. We will assume also that, using this definition of mean quantities, the average of the time derivative of a mean quantity is the time derivative of the mean quantity itself (this is almost true if $T$ is small compared to the mean flow).

## 1.5 Reynolds equations

Substituting decomposition (1.4) into Navier-Stokes equations (1.1), we obtain

$$\begin{cases} \partial_t U_i + \partial_t u'_i + \sum_{j=1}^{N} \left( U_j \partial_j U_i + U_j \partial_j u'_i + u'_j \partial_j U_i + u'_j \partial_j u'_i \right) = \\ = -\frac{1}{\rho}\partial_i P - \frac{1}{\rho}\partial_i p' + \nu \Delta U_i + \nu \Delta u'_i + f_i \qquad i = 1, \dots, N \\ \sum_{j=1}^{N} \partial_j U_j + \sum_{j=1}^{N} \partial_j u'_j = 0 \end{cases} \tag{1.6}$$

and, by averaging them over a set of experimental data or over a long time scale, using average properties and indicating this operation with an overbar, we obtain

$$\begin{cases} \partial_t U_i + \sum_{j=1}^{N} U_j \partial_j U_i - \sum_{j=1}^{N} \partial_j \tau_{ij} = -\frac{1}{\rho}\partial_i P + \nu \Delta U_i + f_i \qquad i = 1, \dots, N \\ \sum_{j=1}^{N} \partial_j U_j = 0. \end{cases} \tag{1.7}$$

The symbol $\tau_{ij} = -\overline{u'_i u'_j}$ is the Reynolds stress tensor which is in perfect analogy with the viscous stress tensor modeled by the assumption of newtonian fluid. This tensor contains six unknown quantities which are a priori not related to the other four mean quantities (three mean velocity components and the mean pressure). Therefore our problem has ten unknowns and only four equations[1] and a closure problem arises: the Reynolds stress tensor needs to be modeled.

## 1.6 Boussinesq's approximation

The oldest proposal for modeling the turbulent or Reynolds stress tensor is a significant part of most turbulence models of practical use today: Boussinesq's (1877) eddy viscosity concept which assumes that, in analogy to the viscous stresses in laminar flows, the turbulent stresses are proportional to the mean velocity gradients. For general flow situations, this concept may be expressed as

$$\tau_{ij} = \nu_t (\partial_j U_i + \partial_i U_j), \tag{1.8}$$

where $\nu_t$ is the turbulent eddy viscosity which, in contrast to the molecular viscosity $\nu$, is not a fluid property but depends strongly on the state of turbulence; $\nu_t$ may vary significantly from one point in the flow to another and also from flow to flow. Therefore the introduction of equations (1.8) alone does not constitute a turbulence model but only provides the framework for constructing such a model: the main problem is now shifted to determine the distribution of $\nu_t$.

The eddy viscosity concept was conceived by presuming an analogy between the molecular motion, which leads to Stokes' viscosity law in laminar flow, and the turbulent motion. The turbulent eddies were thought of as lumps of fluid which, like molecules, collide and exchange momentum. The molecular viscosity is proportional to the average velocity and mean free path of the molecules; accordingly the eddy viscosity is considered proportional to a velocity characterizing the fluctuating motion and to a typical length of this motion which Prandtl called "mixing length". It has often been pointed out [11, 4] that the analogy between molecular and turbulent motion cannot be correct in principle because the turbulent eddies are not rigid bodies which retain their identity and because the large eddies responsible for the momentum transfer are not small compared with the flow domain, as required by the kinetic gas theory. In spite of these objections, the eddy viscosity concept has often been found to work well in practice, simply because $\nu_t$ as defined by equations (1.8) can be determined to a good approximation in many flow

[1] Originally also the continuity equation for turbulent velocity existed, but we used it to simplify the first $N$ equations.

situations. The main success of the eddy viscosity concept was the prediction of two-dimensional thin shear layers.

However, even in categories of relatively simple flows, the eddy viscosity concept sometimes breaks down. In wall jet and asymmetric wall shear layers, like flow in an annulus or channel flows with different wall roughness on either side, regions exist where the stress tensor $\tau_{12}$ component and the velocity gradient $\partial_1 U_2$ have opposite signs, leading to a negative eddy viscosity which is only mathematically possible but not physically meaningful, since in this case turbulence would not be dissipative, but contribute to the mean motion. In flows where complexity is greater than in this shear layers, more than one turbulent stress component is relevant. Equations (1.8) introduce the eddy viscosity $\nu_t$ as a scalar, that is the eddy viscosity is the same for all stress components. This assumption of an isotropic eddy viscosity is a simplification which is of limited realism in complex flows. Therefore, different eddy viscosities are sometimes introduced for the turbulent momentum transport in different directions; for example in large water bodies $\nu_t$ is often prescribed differently for the horizontal and vertical transport. In spite of all the shortcomings of the eddy viscosity concept mentioned above, it has proved successful in many practical calculations and is still the basis of most turbulence models in use today.

### 1.6.1 Generalization of Boussinesq's approximation

We derive here a generalization of relations (1.8), in the three-dimensional case, showing that relations (1.8) come from a much more general relation between the Reynolds stress tensor and the mean velocity deformation tensor $D_{ij} = 1/2(\partial_j U_i + \partial_i U_j)$ with the only assumption that turbulence be isotropic[2].

**Theorem 1** *Let us assume that there exist a linear relation between tensor $D$ and tensor $\tau$, $\tau_{ij} = \sum_{h,k=1}^{3} A_{ijhk} D_{hk}$, then a principal terna[3] for tensor $D$ is principal for tensor $\tau$ too.*

**Proof:**

[2]It is exactly this assumption, which is generally not true, that is the main criticism to Boussinesq's approximation.

[3]A principal terna for a tensor is a system of coordinates where a tensor is diagonal.

In a principal terna for tensor $D$ its off diagonal components are zero and its relation with $\tau$ becomes

$$\tau_{ij} = \sum_{h=1}^{3} A_{ijhh} D_{hh}.$$

Inverting orientation of $x_3$ axis, the components of $D'$ in the new coordinates system do not change since

$$D'_{33} = \frac{\partial U'_3}{\partial x'_3} = \frac{\partial(-U_3)}{\partial x_3}\frac{\partial x_3}{\partial x'_3} = -\frac{\partial(-U_3)}{\partial x_3} = D_{33}.$$

The components of $\tau'$ in the new coordinates system are

$$\tau'_{23} = -\tau_{23} \qquad \tau'_{13} = -\tau_{13} \qquad \tau'_{33} = \tau_{33}.$$

Therefore from

$$\tau'_{23} = \sum_{i=1}^{3} A_{23ii} D_{ii} \qquad \tau_{23} = \sum_{i=1}^{3} A_{23ii} D_{ii}$$

$$\tau'_{13} = \sum_{i=1}^{3} A_{13ii} D_{ii} \qquad \tau_{13} = \sum_{i=1}^{3} A_{13ii} D_{ii}$$

it follows that $\tau_{13}$ and $\tau_{23}$ are zero in a principal terna for $D$. Inverting the $x_2$ axis we can easily show then $\tau_{12}$ is zero too.

□

The following theorem will assume that we are in a principal terna for tensor $D$ (and therefore for tensor $\tau$) but, since both $D$ and $\tau$ are tensors, their components change with the same law when changing coordinate system and therefore the theorem holds for every coordinate system.

**Theorem 2** *If the flow is isotropic, in a principal terna the linear relation between D and $\tau$ can be reduced from nine to two coefficients. If the trace of tensor D is zero (the continuity equation holds for the mean velocity), then the linear relation becomes exactly Boussinesq's approximation (1.8).*

**Proof:** Let us write the relation between $D$ and $\tau$ in a principal terna using $a_{rs} = A_{rrss}$:

$$\begin{cases} \tau_{11} = a_{11}D_{11} + a_{12}D_{22} + a_{13}D_{33} \\ \tau_{22} = a_{21}D_{11} + a_{22}D_{22} + a_{23}D_{33} \\ \tau_{33} = a_{31}D_{11} + a_{32}D_{22} + a_{33}D_{33}. \end{cases} \tag{1.9}$$

Let us do the following permutation of axis:

$$x_1' = x_3 \qquad x_2' = x_1 \qquad x_3' = x_2.$$

The new components of the two tensors in the new coordinate system are

$$\tau_{11}' = \tau_{33} \qquad \tau_{22}' = \tau_{11} \qquad \tau_{33}' = \tau_{22}$$

$$D_{11}' = D_{33} \qquad D_{22}' = D_{11} \qquad D_{33}' = D_{22}$$

and therefore thanks to (1.9)

$$\begin{cases} \tau_{22}' = \tau_{11} = a_{11}D_{22}' + a_{12}D_{33}' + a_{13}D_{11}' \\ \tau_{33}' = \tau_{22} = a_{21}D_{22}' + a_{22}D_{33}' + a_{23}D_{11}' \\ \tau_{11}' = \tau_{33} = a_{31}D_{22}' + a_{32}D_{33}' + a_{33}D_{11}'. \end{cases}$$

Using now the isotropy hypothesis and comparing the two systems,

$$\begin{cases} a_{11} = a_{33} & a_{12} = a_{31} & a_{13} = a_{32} \\ a_{21} = a_{13} & a_{22} = a_{11} & a_{23} = a_{12} \\ a_{31} = a_{23} & a_{32} = a_{21} & a_{33} = a_{22}, \end{cases}$$

we get only three independent coefficients. Doing an analogous axis permutation

$$x_1' = x_2 \qquad x_2' = x_1 \qquad x_3' = x_3$$

and setting

$$\Lambda = a_{12} = a_{31} = a_{23} = a_{13} = a_{32} = a_{21}$$

$$\mu = -\frac{a_{11} + \Lambda}{2} = -\frac{a_{22} + \Lambda}{2} = -\frac{a_{33} + \Lambda}{2},$$

from (1.9) we can obtain

$$\begin{cases} \tau_{11} = -(\Lambda + 2\mu)D_{11} - \Lambda(D_{22} + D_{33}) \\ \tau_{22} = -(\Lambda + 2\mu)D_{22} - \Lambda(D_{11} + D_{33}) \\ \tau_{33} = -(\Lambda + 2\mu)D_{33} - \Lambda(D_{11} + D_{22}). \end{cases} \tag{1.10}$$

Now, if continuity equation for mean velocity holds (see (1.7)), the trace of tensor $D$ is zero and therefore relations (1.10) becomes Boussinesq's approximation.

□

### 1.6.2 Normal stresses correction

We can immediately see that equations (1.8) are not applicable to normal stresses. In fact, considering $i = j$ and summing over index $j$, we obtain twice the opposite of turbulent kinetic energy

$$-\sum_{j=1}^{N} \overline{u_j'^2} = \sum_{j=1}^{N} \tau_{jj} = 2\nu_t \operatorname{div} U \tag{1.11}$$

which is zero due to the continuity equation for the mean velocity. Therefore turbulent kinetic energy is zero and there is no turbulence. To avoid such paradox, we can modify (1.8) to

$$\tau_{ij} = \nu_t \left( \partial_j U_i + \partial_i U_j \right) - \frac{2}{3} k \delta_{ij}, \tag{1.12}$$

where $k = \frac{1}{2}\sum_{j=1}^{N} \overline{u_j'^2}$ is the turbulent kinetic energy (per unit mass) and $\delta$ is the Kronecker's delta[4]. In this way normal stresses act like pressure forces (i.e. they are perpendicular to the faces of a control volume) and since turbulent energy $k$ is a scalar quantity, it can be absorbed in the pressure term. The static pressure is thus replaced as unknown quantity by the pressure $P + \frac{2}{3}k$. The appearance of $k$ does not necessitate its determination and only the distribution of $\nu_t$ remains to be determined.

## 1.7 Eddy viscosity turbulence models

As we have seen in the previous section, using Boussinesq's approximation the only thing left to be determined is eddy viscosity coefficient $\nu_t$. This coefficient is not constant and depends on the flow; therefore a model where it is considered constant cannot be considered a turbulence model and its only effect is to increase kinematic viscosity. The easiest turbulence model is to prescribe it using an algebraic formula usually involving the product of a scale velocity and a mixing length. However, solutions of one or even two differential equations involving turbulent quantities are used to determine its value too.

[4] $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij}=0$ for $i \neq j$.

### 1.7.1 Zero equations models

The first model to describe the distribution of eddy viscosity was suggested by Prandtl [36] in 1925 and is known as Prandtl mixing length hypothesis. Stimulated by kinetic gas theory, Prandtl assumed that $\nu_t$ is proportional to the product of a mean fluctuating velocity $\tilde{V}$ (scale velocity) and a mixing length $l_m$ (scale length). Considering shear layers with only one significant turbulent stress $\overline{u_1' u_2'}$, where $x_1$ has the same direction as the main flow, and with one velocity gradient $\partial_2 U_1$, he postulated that $\tilde{V}$ is equal to the mean velocity gradient times the mixing length. The mixing length is defined in an experimental way as a not decreasing function of the distance from the boundary; this function depends also on the type of flow. Many further modifications have been done to this model to take into account buoyancy effects and to give a general formula for the mixing length.

In 1942 Prandtl [37] proposed a simpler model applicable only to free shear layers. In this model he assumed eddy viscosity to be constant over any cross section of the layer, the mixing length to be proportional to the layer width and the velocity scale to be proportional to the maximum velocity difference across the layer, with a constant of proportionality depending on the type of flow. This model is quite popular for the prediction of mixing layers, jets and wakes. It works well when these flows are in a developed state, but transitions from one type of free flow to another one are not well predicted.

### 1.7.2 One equation models

In order to overcome the limitations of the mixing length hypothesis, turbulence models were developed which account for the transport of turbulence quantities by solving differential transport equations for them. An important step in the development was to give up the direct link between the fluctuating velocity scale and the mean velocity gradients and to determine this scale from a transport equation.

If the velocity fluctuations have to be characterized by one scale, the physically most meaningful scale is $\sqrt{k}$, where $k$ is the kinetic energy of the turbulent motion (per unit mass)

$$\nu_t = c_\mu \sqrt{k} l_m. \tag{1.13}$$

This formula is known as the Kolmogorov-Prandtl expression, where kinetic energy $k$ is determined by solving a transport equation

$$\partial_t k + \sum_{j=1}^{N} U_j \partial_j k = \sum_{j=1}^{N} \partial_j \left[ \left( \nu + \frac{\nu_t}{\sigma_k} \right) \partial_j k \right] + \tag{1.14}$$

$$+\nu_t \sum_{i,j=1}^{N} (\partial_i U_j + \partial_j U_i) \partial_j U_i - \epsilon_t.$$

Here $\epsilon_t$ is the turbulent energy dissipation, which can be approximated by $c_D k^{\frac{3}{2}} / l_m$, while $c_\mu$, $c_D$ and $\sigma_k$ are empirical constants. This equation is first derived exactly from Navier-Stokes equations and then approximated to avoid including further unknowns.

### 1.7.3 Two equations models

The length scale $l_m$ characterizing the size of the large energy-containing eddies is subject to transport processes in a similar manner as the energy $k$. Other processes influencing the length scale are dissipation, which destroys the small eddies and thus effectively increases the eddy size, and vortex stretching connected with the energy cascade, which reduces the eddy size. The balance of all these processes can be expressed in a transport equation for $l_m$ which can then be used to calculate its distribution. The difficulties in finding widely valid formulae for prescribing or calculating $l_m$ have stimulated the use of such a length scale equation.

A length scale equation does not necessarily need the mixing length itself as dependent variable; any combination of $k$ and $l_m$ will suffice because $k$ is known from solving the energy equation (1.14). The most popular one is called the $k$-$\epsilon$ model and was suggested by Chou [8] involving the turbulent energy dissipation $\epsilon_t$

$$\begin{aligned} \partial_t \epsilon_t + \sum_{j=1}^{N} U_j \partial_j \epsilon_t = \sum_{j=1}^{N} \partial_j \left[ \left( \nu + \frac{\nu_t}{\sigma_\epsilon} \right) \partial_j \epsilon_t \right] + \\ + c_{1\epsilon} \nu_t \frac{\epsilon_t}{k} \sum_{j,i=1}^{N} (\partial_j U_i + \partial_i U_j) \partial_j U_i - c_{2\epsilon} \frac{\epsilon_t^2}{k}, \end{aligned} \tag{1.15}$$

where $c_{1\epsilon}$, $c_{2\epsilon}$ and $\sigma_\epsilon$ are empirical constants. This equation is derived exactly from equations for the fluctuating vorticity [28] and then drastically modeled to avoid introducing further unknowns.

## 1.8 Large eddy simulation models

The main difference between large eddy simulation (LES) turbulence models and models using Reynolds equations is the averaging procedure performed on Navier-Stokes equations. The LES technique does not involve the use of experimental averages or time scale averages as a first step in obtaining equations for the mean flow. Rather, a space filtering operation involving Fourier transforms is applied to the equations of motion. We will not go into deep details which can be found in Yeo [43] or Aldama [1].

### 1.8.1 The filtering technique

Let $f(x,t)$ be an instantaneous variable (velocity or pressure) which appears in Navier-Stokes equations (1.1). Its corresponding filtered variable is defined by the convolution integral

$$F(x,t) = \overline{f}(x,t) = \int_{\mathbb{R}^N} H(x-\xi) f(\xi,t)\, d\xi \tag{1.16}$$

where $H$ is a suitably defined filter function. The effect of the filtering operation becomes clear by taking the Fourier transform of expression (1.16). By definition, the space Fourier transform of f is given by

$$\widehat{f}(h,t) = \int_{\mathbb{R}^N} f(x,t) e^{-\mathrm{i} h \cdot x}\, dx \tag{1.17}$$

where $h$ represent the wave number vector and $\mathrm{i} = \sqrt{-1}$. Thus, by the convolution theorem, we get

$$\widehat{\overline{f}}(h,t) = \widehat{H}(h)\widehat{f}(h,t). \tag{1.18}$$

If $\widehat{H} = 0$ for $|h_i| > h_C$, where $h_C$ is a cut-off wave number, all the high wave number components of $f$ are filtered out by convoluting if with $H$. A filter with such characteristics is denoted by Holloway [19] an *ideal low pass filter*. However, if the filter function in wave number space $\widehat{H}$ rapidly falls off, a cut-off wave number can also be defined for all practical purposes. Most commonly box filters and Gaussian filters have also been considered by Ferzinger [14].

We can represent any filter with the following expression

$$H(x) = \prod_{j=1}^{N} H_j(x_j) \tag{1.19}$$

where $H_j(x_j)$ is a one-dimensional filter. From (1.17) and (1.19) the Fourier transform of $H$ can be written as

$$\widehat{H}(h) = \prod_{j=1}^{N} \widehat{H}_j(h_j) \tag{1.20}$$

where $\widehat{H}_j(k_j)$ represent the Fourier transform of $H_j(x_j)$ with respect to $x_j$ and is defined by

$$\widehat{H}_j(h_j) = \int_{\mathbb{R}} H_j(x_j) e^{-\mathrm{i} h_j x_j} \, dx_j \qquad j = 1, \ldots, N. \tag{1.21}$$

If an ideal low pass filter is used

$$H_j(x_j) = \frac{\sin \frac{2\pi x_j}{\lambda}}{\pi x_j} \tag{1.22}$$

$$\widehat{H}_j(h_j) = \begin{cases} 1 & \text{for} \quad |h_j| \leq \frac{2\pi}{\lambda} \\ 0 & \text{for} \quad |h_j| > \frac{2\pi}{\lambda} \end{cases}, \tag{1.23}$$

while for the box filter we have

$$K_j(x_j) = \begin{cases} \frac{1}{\lambda} & \text{for} \quad |x_j| \leq \frac{\lambda}{2} \\ 0 & \text{for} \quad |x_j| > \frac{\lambda}{2} \end{cases}, \tag{1.24}$$

$$\widehat{K}_j(h_j) = \frac{\sin \frac{\lambda h_j}{2}}{\lambda h_j} \tag{1.25}$$

and for the Gaussian filter

$$G_j(x_j) = \sqrt{\frac{\gamma}{\pi \lambda^2}} \exp\Big(-\gamma \frac{x_j^2}{\lambda^2}\Big) \tag{1.26}$$

$$\widehat{G}_j(h_j) = \exp\Big(-\frac{\lambda^2 h_j^2}{4\gamma}\Big). \tag{1.27}$$

In expressions (1.22)–(1.27) $\lambda$ represents a characteristic filter width and in (1.26) and (1.27) $\gamma$ is a parameter usually set equal to 6 for reasons explained later on. It can be observed that a clear cut-off wave number equal to $2\pi/\lambda$ can be defined for the ideal low pass filter. In contrast, the Fourier transform of the box filter is a dumped sinusoid and spurious amplitude reversals are produced by its use in Fourier space. Finally, the Fourier transform of a

Gaussian filter is also Gaussian and for all practical purposes it is essentially contained in the range $[-2\pi/\lambda, 2\pi/\lambda]$.

Based on the previous discussion, we conclude that a filtering operation such as the one defined by (1.16) tends to eliminate from the filtered variables the rapidly space-fluctuating components, usually characterized as *turbulence*. It can also be shown that the filter operator commute with the spatial and temporal derivatives [25]. Thus, the filtered Navier-Stokes equations are

$$\begin{cases} \partial_t U_i + \sum_{j=1}^{N} \partial_j \overline{u_i u_j} = -\frac{1}{\rho}\partial_i P + \nu \Delta U + f_i & i = 1, \dots, N \\ \sum_{j=1}^{N} \partial_j u_j = 0, \end{cases} \tag{1.28}$$

The instantaneous velocity field can be decomposed as usual as

$$u_i = U_i + u_i' \tag{1.29}$$

where $u_i'$ represents the high wave number component of the velocity field. In LES literature, $U$ is called the Large Scale (LS) velocity and $u'$ the Sub Grid Scale (SGS) velocity. When this decomposition is used, the following result is obtained

$$\overline{u_i u_j} = \overline{U_i U_j} + \overline{U_i u_j'} + \overline{u_i' U_j} + \overline{u_i' u_j'}. \tag{1.30}$$

For a general space filtering operation the classical Reynolds postulate $\overline{U_i} = U_i$ does not apply and therefore relation (1.30) cannot be further simplified and system (1.28) results as an integro-differential not closed systems of equations.

### 1.8.2 Leonard model

In 1974 Leonard [25] proposed that the LS advective term $U_i U_j$ be expanded in terms of a Taylor series inside the convolution integral that defines its filtered value. The resulting integrals do not converge for the ideal low pass filter; however, for the Gaussian filter Leonard approximation is

$$\overline{U_i U_j} = U_i U_j + \frac{\lambda^2}{4\gamma}\Delta(U_i U_j) + O(\lambda^4). \tag{1.31}$$

The choice of $\gamma$ is arbitrary, but (1.31) coincides with the box filter approximation when $\gamma = 6$. Because $U_i U_j$ is already a smooth function in the scale of the filter width, its expansion in terms of a Taylor series seems to be justified. Through numerical experimentation Kwak at al. [22] have shown that for isotropic Cartesian meshes it is appropriate to take $\lambda$ equal to twice the computational grid size. This result has significant theoretical appeal, as the size of the filter width coincides with the characteristic length scale of the smallest resolvable eddies.

However, the problem with Leonard's approximation is that the LS advective terms appears inside a derivative in the filtered governing equations (1.28); therefore a system of third-order differential equations results. This fact raises questions about the well-posedness and, from practical standpoint, causes problems at the boundary due to the lack of boundary conditions.

### 1.8.3 Clark et al.'s model

In 1977 Clark et al. [9] pursued the idea of expanding both $U$ and $u'$ in a Taylor series in the convolution integral that defines the cross term $\overline{U_i u'_j}$. However, the use of Taylor series expansion of $u'$ is not rigorous, since the turbulent component $u'$ is highly fluctuating in the scale of the filter width.

To overcome this problem, Aldama noticed that formula (1.31) for Leonard's approximation can be obtained using the Fourier transform of the Gaussian filter (1.27) and expanding it in Taylor series as

$$\widehat{G}(h) = 1 - \frac{\lambda^2}{4\gamma} h^2 + O(\lambda^4), \tag{1.32}$$

substituting it into (1.18)

$$\widehat{\overline{U_i U_j}}(h) = \widehat{U_i U_j} - \frac{\lambda^2}{4\gamma} h^2 \widehat{U_i U_j}(h) + O(\lambda^4) \tag{1.33}$$

and finally using the property of the Fourier transform

$$-h_l^2 \widehat{U_i U_j}(h) = [\partial_l \partial_l \widehat{(U_i U_j)}](h) \tag{1.34}$$

and applying the inverse Fourier transform to reduce (1.33) to (1.31).

In the same way we can obtain an approximation of the cross terms $\overline{U_i u'_j}$ and $\overline{u'_i U_j}$. Their Fourier transform can be written as

$$\widehat{\overline{U_i u'_j}}(h) = \widehat{G}(h) \int_{\mathbb{R}^N} \widehat{U_i}(h - \xi) \widehat{u'_j}(\xi)\, d\xi. \tag{1.35}$$

Since $\widehat{G}(h) \neq 0$ for every $h$, using (1.18) and (1.29) we obtain

$$\widehat{u'_j}(\xi) = \left[\frac{1}{\widehat{G}(\xi)} - 1\right]\widehat{U_j}(\xi). \tag{1.36}$$

Substituting it into (1.35) and expanding $\widehat{G}(h)$ and $\widehat{G}(\xi)$ in Taylor series gives

$$\widehat{\overline{U_i u'_j}}(h) = \int_{\mathbb{R}^N} \widehat{U_i}(h-\xi)\widehat{U_j}(\xi)\left[\frac{\lambda^2}{4\gamma}\xi^2 + O(\lambda^4)\right] d\xi. \tag{1.37}$$

In analogy to (1.34)

$$-\xi_l^2\widehat{U_j}(\xi) = [\widehat{\partial_l\partial_l U_j}](\xi), \tag{1.38}$$

substituting (1.38) into (1.37) and taking the inverse Fourier transform gives a formal series approximation of $\overline{U_i u'_j}$

$$\overline{U_i u'_j} = -\frac{\lambda^2}{4\gamma}U_i \Delta U_j + O(\lambda^4). \tag{1.39}$$

This equation is identical to the approximation obtained by Clark at al.. According to Aldama [1] the convergence of the series generating the Leonard approximation and the approximation of the cross terms cannot be proved because convergence tests are inconclusive in these cases. Nevertheless, proving convergence is not needed for those approximations to be useful. Demonstrating their asymptotic nature would suffice and, in fact, would be even better than establishing their convergence. Indeed, in the approximation of functions, truncated asymptotic series are known to give better numerical approximation than truncated convergent series in a wide variety of cases [3, 34], even when the former diverge. But the most important implication of such a proof would be that a formal perturbation theory can be built around asymptotic approximation. As a consequence, a measure of the size of the error made in truncating asymptotic series can be given. Aldama [1, Section 3 and 4] gives a proof of the asymptotic nature of the series generating the Leonard approximation and also a proof for the case of the approximation of the cross terms.

The main advantage of this treatment is that now combining (1.39) and (1.31) together we obtain

$$\overline{u_i u_j} = U_i U_j + \frac{\lambda^2}{2\gamma}\sum_{l=1}^{N} \partial_l U_i \partial_l U_j + O(\lambda^4) + \overline{u'_i u'_j}, \tag{1.40}$$

where the second order derivatives no longer appear. Therefore the space filtered flow equations, given a proper closure model for $\overline{u_i'u_j'}$, are second-order differential equations.

### 1.8.4 Yeo and Bedford's model

Following an approach of Yeo and Bedford, Cantekin and Westerink [7], expanded the SGS terms $\overline{u_i'u_j'}$ in the same way as the cross terms. In analogy with (1.35) we get

$$\widehat{\overline{u_i'u_j'}}(h) = \widehat{G}(h) \int_{\mathbb{R}^N} \widehat{u_i'}(h-\xi)\widehat{u_j'}(\xi)\, d\xi, \tag{1.41}$$

using (1.36) and expanding the Fourier transform of Gaussian filters in Taylor series we obtain

$$\widehat{\overline{u_i'u_j'}}(h) = \int_{\mathbb{R}^N} \Big( \frac{\lambda^4}{16\gamma^2} h^2 (h-\xi)^2 + O(\lambda^6) \Big) \widehat{U_i}(h-\xi)\widehat{U_j}(\xi)\, d\xi \tag{1.42}$$

and therefore the series approximation of the SGS terms is

$$\overline{u_i'u_j'} = \frac{\lambda^4}{16\gamma^2} \sum_{l=1}^{N} (\partial_l \partial_l U_i)(\partial_l \partial_l U_j) + O(\lambda^6). \tag{1.43}$$

We note that the leading terms in this approximation are of $O(\lambda^4)$ and will therefore be neglected in the flow equations (1.28). A similar approximation to the SGS terms was first obtained by Yeo and Bedford [44, 2]. Using this model, no closure model is required but the drawback is that we formally neglect the contribution of the SGS terms, which are considered as one of the main source of turbulent effects.

*Space filtered Navier-Stokes (SF) equations* are therefore

$$\begin{cases} \partial_t U_i + \sum_{j=1}^{N} U_j \partial_j U_i = -\frac{1}{\rho}\partial_i P + \nu \Delta U_i + \\ \qquad - \sum_{j,l=1}^{N} \partial_j \Big[ \frac{\lambda^2}{2\gamma} \partial_l U_i \partial_l U_j \Big] + f_i \qquad i = 1, \ldots, N \\ \sum_{j=1}^{N} \partial_j U_j = 0, \end{cases} \tag{1.44}$$

which, together with boundary conditions $U_{|\partial\Omega}= 0$ and initial conditions $U_{|t=0}= U_0$ give the space filtered Navier-Stokes model. Clearly, the initial datum is required to satisfy

$$\sum_{j=1}^{N} \partial_j U_{0j} = 0 \qquad\qquad U_{0|\partial\Omega} = 0. \tag{1.45}$$

We will study these equations in Chapter 3, where we will give a global existence and uniqueness theorem for the analytical solution provided initial data are regular and small enough. In Chapter 5 we will study the numerical performance of this model.

### 1.8.5 Smagorinsky's model

The most popular form of closure model for modeling $\overline{u_i'u_j'}$ is the Smagorinsky's model [42]

$$-\overline{u_i'u_j'} = c\lambda^2\|D\|D_{ij} - \frac{1}{3}k\delta_{ij}, \tag{1.46}$$

where $D$ is the averaged velocity deformation tensor and $k$ the turbulent kinetic energy. This model has been used by Findikakis and Street [15] to simulate uniform density and thermally stratified steady-state turbulent flow in a lid driven cavity, by Clark at al. [9] to simulate the decay of homogeneous grid-generated turbulence, by Dakhoul and Debford [12] to simulate the quasi-turbulence associated with the one-dimensional Burgers' equation. However many other authors simply approximate the term $\overline{u_iu_j}$ with $U_iU_j$ plus a Smagorinsky's model for the remaining terms [35, 45].

We will call SF model with Smagorinsky's closure *eddy viscosity space filtered (EVSF) model*

$$\begin{cases} \partial_t U_i + \sum_{j=1}^{N} U_j\partial_j U_i = -\frac{1}{\rho}\partial_i P + \sum_{j=1}^{N}\partial_j\Big[(\nu + C\|\nabla U\|^{2\mu})\partial_j U_i\Big] + \\ \qquad\qquad - \sum_{j,l=1}^{N}\partial_j\Big[\frac{\lambda^2}{2\gamma}\partial_l U_i \partial_l U_j\Big] + f_i \qquad i = 1,\dots,N \\ \sum_{j=1}^{N}\partial_j U_j = 0,. \end{cases} \tag{1.47}$$

and we will show in Chapter 4 a global existence and uniqueness theorem for its analytical solution and in Chapter 5 some numerical approximation results. The equations obtained by the Smagorinsky's model dropping the LS advective terms will be simply called *eddy viscosity (EV) model* and we will provide numerical solutions for it in Chapter 5, while existence and uniqueness theorem for its analytical solution can be found in [23].

# Chapter 2

# $k$-$\epsilon$ and $\varphi$-$\theta$ models

Although the $k$-$\epsilon$ two-equations turbulence model is the most widely used turbulence model by engineers and scientists, its mathematical properties remains to be clarified. There are a lot of numerical and physical experiments about the $k$-$\epsilon$ model but very few mathematical studies. In this chapter we are going to introduce another equivalent model, the $\varphi$-$\theta$ , and we are going to show positivity, existence and uniqueness of its analytical solution following a work by Mohammadi [32] on compressible flows. We then propose a numerical approximation of $\varphi$-$\theta$ model which leads to a stable positive solution for both the spatially discretized problem (semidiscretization) and for the temporal-spatially discretized problem (total discretization).

## 2.1 Construction of $\varphi$-$\theta$ model

The $\varphi$-$\theta$ model is built from $k$-$\epsilon$ model changing the variables in such a way that in the new two differential equations we eliminate dissipative terms which depends with positive powers on the other equation's variable. This is clearly done to avoid having dissipative terms such as the $-\epsilon_t$ in equation (1.14) which is the term which causes the most critical numerical problems.

We start from $k$-$\epsilon$ model (1.14) (1.15) in its version for compressible fluids

$$\begin{cases} \nu_t = c_\mu \dfrac{k^2}{\epsilon_t} \\ d_t k = \sum\limits_{j=1}^{N} \partial_j \left[ \left( \nu + \dfrac{\nu_t}{\sigma_k} \right) \partial_j k \right] + c_\mu \dfrac{k^2}{\epsilon_t} F - \dfrac{2}{3} k \operatorname{div} U - \epsilon_t \\ d_t \epsilon_t = \sum\limits_{j=1}^{N} \partial_j \left[ \left( \nu + \dfrac{\nu_t}{\sigma_\epsilon} \right) \partial_j \epsilon_t \right] + c_1 k F - \dfrac{2}{3} \dfrac{c_1}{c_\mu} \epsilon_t \operatorname{div} U - c_2 \dfrac{\epsilon_t^2}{k} \end{cases} \tag{2.1}$$

where $c_1 = c_{1\epsilon} \cdot c_\mu \approx 0.1296$, $c_2 = c_{2\epsilon} \approx 1.92$, $c_\mu \approx 0.09$, $\sigma_k \approx 1$, $\sigma_\epsilon \approx 1.3$ and

$$d_t = \partial_t + \sum_{j=1}^{N} U_j \partial_j \tag{2.2}$$

$$F = \sum_{i,j=1}^{N} (\partial_i U_j + \partial_j U_i) \partial_j U_i - \frac{2}{3} (\operatorname{div} U)^2. \tag{2.3}$$

The last term of (2.2) is called *convective term*, the second term of differential equations (2.1) *diffusive term*, the third *production term*, the fourth *compressibility term* and the fifth *dissipative term*.

We now introduce $\theta$ and $\varphi$ variables using the fact that $k$ and $\epsilon_t$, for physical reasons (they are an energy and a dissipation), should always be strictly positive in a turbulent flow. We set

$$\theta = \frac{k}{\epsilon_t} \qquad \varphi = \frac{\epsilon_t^2}{k^3} \tag{2.4}$$

so that

$$k = \frac{1}{\theta^2 \varphi} \qquad \epsilon_t = \frac{1}{\theta^3 \varphi}.$$

Let us write now differential equations (2.1) in the new variables

$$-\frac{1}{\theta^2 \varphi^2} \partial_t \varphi - \frac{2}{\theta^3 \varphi} \partial_t \theta - \frac{1}{\theta^2 \varphi^2} \sum_{j=1}^{N} U_j \partial_j \varphi - \frac{2}{\theta^3 \varphi} \sum_{j=1}^{N} U_j \partial_j \theta =$$

$$= \mathrm{Diff}_k + c_\mu F \frac{1}{\theta \varphi} - \frac{2}{3} \frac{1}{\theta^2 \varphi} \operatorname{div} U - \frac{1}{\theta^3 \varphi}, \tag{2.5}$$

$$-\frac{1}{\theta^3 \varphi^2} \partial_t \varphi - \frac{3}{\theta^4 \varphi} \partial_t \theta - \frac{1}{\theta^3 \varphi^2} \sum_{j=1}^{N} U_j \partial_j \varphi - \frac{3}{\theta^4 \varphi} \sum_{j=1}^{N} U_j \partial_j \theta =$$

$$= \mathrm{Diff}_\epsilon + c_1 F\frac{1}{\theta^2\varphi} - \frac{2}{3}\frac{c_1}{c_\mu}\frac{1}{\theta^3\varphi}\mathrm{div}\, U - c_2\frac{1}{\theta^4\varphi}. \tag{2.6}$$

where terms $\mathrm{Diff}_k$ and $\mathrm{Diff}_\epsilon$ are the diffusive terms of $k$-$\epsilon$ model expressed through the new variables $\varphi$ and $\theta$.

Multiplying equation (2.6) by $\theta$, subtracting the result from equation (2.5) and multiplying it by $\theta^3\varphi$ we obtain the $\theta$-equation

$$d_t\theta = \mathrm{Diff}_\theta + (c_\mu - c_1)F\theta^2 + \frac{2}{3}\theta\Big(\frac{c_1}{c_\mu} - 1\Big)\mathrm{div}\, U + (c_2 - 1). \tag{2.7}$$

Subtracting three times equation (2.5) from twice equation (2.6) multiplied by $\theta$ and multiplying the result by $\theta^2\varphi^2$ we obtain the $\varphi$-equation

$$d_t\varphi = \mathrm{Diff}_\varphi + (2c_1 - 3c_\mu)F\theta\varphi + \frac{2}{3}\varphi\Big(3 - 2\frac{c_1}{c_\mu}\Big)\mathrm{div}\, U + (3 - 2c_2)\frac{\varphi}{\theta}. \tag{2.8}$$

Diffusive terms $\mathrm{Diff}_\theta$ and $\mathrm{Diff}_\varphi$ are

$$\mathrm{Diff}_\theta = \sum_{j=1}^{N}\partial_j(\nu + c_\theta\nu_t)\varphi\theta + G_\theta$$

$$\mathrm{Diff}_\varphi = \sum_{j=1}^{N}\partial_j(\nu + c_\varphi\nu_t)\partial_j\varphi + G_\varphi$$

$$\begin{aligned} G_\theta = {} & \Big(\frac{3\sigma_k - 2\sigma_\epsilon}{\sigma_k\sigma_\epsilon} - c_\theta\Big)\nu_t\Delta\theta - \Big(\frac{15\sigma_k - 8\sigma_\epsilon}{\sigma_k\sigma_\epsilon} - c_\theta\Big)\frac{\nu_t}{\theta}\nabla\theta\cdot\nabla\theta + \\ & -\Big(\frac{10\sigma_k - 7\sigma_\epsilon}{\sigma_k\sigma_\epsilon} - c_\theta\Big)\frac{\nu_t}{\varphi}\nabla\theta\cdot\nabla\varphi + \frac{\sigma_k - \sigma_\epsilon}{\sigma_k\sigma_\epsilon}\nu_t\Big[\frac{\theta}{\varphi}\Delta\varphi + \\ & -3\frac{\theta}{\varphi^2}\nabla\varphi\cdot\nabla\varphi\Big] - 6\nu\frac{1}{\theta}\nabla\theta\cdot\nabla\theta - 2\nu\frac{1}{\varphi}\nabla\varphi\cdot\nabla\varphi \end{aligned}$$

$$\begin{aligned} G_\varphi = {} & \Big(\frac{3\sigma_k - 2\sigma_\epsilon}{\sigma_k\sigma_\epsilon} - c_\varphi\Big)\nu_t\Delta\theta - \Big(\frac{21\sigma_\epsilon - 20\sigma_k}{\sigma_k\sigma_\epsilon} - c_\varphi\Big)\frac{\nu_t}{\theta}\nabla\varphi\cdot\nabla\theta + \\ & -\Big(\frac{3\sigma_\epsilon - 2\sigma_k}{\sigma_k\sigma_\epsilon} - c_\varphi\Big)\frac{\nu_t}{\varphi}\nabla\varphi\cdot\nabla\varphi + 6\frac{\sigma_k - \sigma_\epsilon}{\sigma_k\sigma_\epsilon}\nu_t\frac{\varphi}{\theta}\Delta\theta \\ & +6\frac{5\sigma_k - 4\sigma_\epsilon}{\sigma_k\sigma_\epsilon}\nu_t\frac{\varphi}{\theta}\nabla\theta\cdot\nabla\theta - 2\nu\frac{1}{\varphi}\nabla\varphi\cdot\nabla\varphi + 6\nu\frac{\varphi}{\theta^2}\nabla\theta\cdot\nabla\theta. \end{aligned}$$

Note that these diffusive terms are more general than those considered by Mohammadi in [32], where the author takes $\sigma_k = \sigma_\epsilon$, neglects the fifth term of $G_\varphi$ and takes $\nu = 0$. While the first two hypothesis can hold, considering kinematic viscosity to be zero can radically change the mathematical nature of the problem. In fact to show existence and uniqueness, Mohammadi turns again to $\nu > 0$, even if in [33, Appendix A5] Lewandowski shows existence, positivity and uniqueness theorems even for $\nu = 0$ but only in the incompressible case.

In [32] Mohammadi says that constants $c_\theta$ and $c_\varphi$ need to be numerically tuned, but we stress the fact that in any case there is not a value of $c_\theta$, $c_\varphi$, $\nu$, $\sigma_k$ and $\sigma_\epsilon$ which let $G_\theta$ and $G_\varphi$ vanish. This and the former observation imply that the $\varphi$-$\theta$ model we are going to use is not the $k$-$\epsilon$ model in other variables, but a modeled version which assumes that $\mathrm{Diff}_\theta$ and $\mathrm{Diff}_\varphi$ are like diffusive terms in $k$-$\epsilon$ model and through use of $c_\theta$ and $c_\varphi$ tries to have $G_\theta \approx 0$ and $G_\varphi \approx 0$.

We decided to follow Mohammadi and using $\varphi$-$\theta$ model instead of $k$-$\epsilon$ model because the former has much better numerical properties than the latter. Especially interesting are the positivity numerical results for $\varphi$-$\theta$ model which are much more difficult to obtain for the $k$-$\epsilon$ model. From an analytical point of view, positivity for $\varphi$-$\theta$ model is proven in this chapter thanks to Mohammadi while some partial positivity results, applicable only when the solution is very regular, for $k$-$\epsilon$ model are in [32, Section II.3], [33, Chapter 5, section 2.3] and [33, Chapter 9, section 2].

## 2.2 Gronwall lemmae

We give here some versions of the Gronwall lemma, which will be used many times in this work. A proof of these lemmae can be found in [38, Chapter 1, Section 4].

**Lemma 1 (Gronwall)** *Let $G(t)$ be a non-decreasing function and $\alpha(t)$ and $\psi(t)$ be non-negative functions. If*

$$\psi(t) \leq G(t) + \int_0^t \alpha(s)\psi(s)\,\mathrm{ds}$$

*then*

$$\psi(t) \leq G(t) \exp\left[\int_0^t \alpha(s)\,\mathrm{ds}\right].$$

**Observation 1** *Gronwall lemma can also be formulated as: let $g(t)$, $\alpha(t)$ and $\psi(t)$ be non-negative functions. If*

$$\partial_t\psi(t) \leq g(t) + \alpha(t)\psi(t)$$

*then*

$$\psi(t) \leq \left[\int_0^t g(s)\,\mathrm{ds} + \psi(0)\right] \exp\left[\int_0^t \alpha(s)\,\mathrm{ds}\right].$$

**Lemma 2 (Discrete Gronwall)** *Let $K_n$ be a non-negative sequence and $\alpha$ a constant not smaller than $\psi_0$. If*

$$\psi_n \leq \alpha + \sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} K_s\psi_s \qquad \forall\, n \geq 1,$$

*then*

$$\psi_1 \leq \alpha(1 + K_)) + p_0$$

$$\psi_n \leq \alpha \prod_{s=0}^{n-1}(1 + K_s) + \sum_{s=0}^{n-2} p_s \prod_{r=s+1}^{n-1} (1 + K_r) + p_{n-1}$$

**Observation 2** *If in discrete Gronwall lemma 2 succession $p_n$ is non-negative, then the thesis can be changed in*

$$\psi_n \leq \left(\alpha + \sum_{s=0}^{n-1} p_s\right) \exp\left(\sum_{s=0}^{n-1} K_s\right).$$

## 2.3 Solution of $\varphi$-$\theta$ model

In this section we are going to follow Mohammadi [32] and show that if

$$U \in L^\infty(0,T;L^\infty(\Omega)), \quad \operatorname{div} U \in L^2(0,T;L^\infty(\Omega)),$$
$$F \in L^2(0,T;L^\infty(\Omega)) \tag{2.9}$$

then there exist a non-negative unique solution of $\varphi$-$\theta$ reduced model (2.13) in $L^\infty(0,T;L^\infty(\Omega) \cap H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$. This result will be used to prove that if

$$U \in L^\infty(0,T;L^\infty(\Omega)), \quad \operatorname{div} U \in L^\infty(0,T;L^\infty(\Omega)),$$
$$F \in L^\infty(0,T;L^\infty(\Omega)) \tag{2.10}$$

then there exist in $L^\infty(0,T;L^\infty(\Omega)) \cap L^2(0,T;H^1(\Omega))$ a positive unique solution of $\varphi$-$\theta$ problem (2.11).

### 2.3.1 The $\varphi$-$\theta$ problem

The $\varphi$-$\theta$ problem is stated as follows

$$\begin{cases} \nu_t = \dfrac{c_\mu}{\theta\varphi} \\ d_t\theta = \sum_{j=1}^{N} \partial_j[(\nu + c_\theta\nu_t)\partial_j\theta] - c_3F\theta^2 + c_4\theta \operatorname{div} U + c_5 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall x \in \Omega \quad \forall t \in [0,T] \\ d_t\varphi = \sum_{j=1}^{N} \partial_j[(\nu + c_\varphi\nu_t)\partial_j\varphi] - c_6F\theta\varphi + c_7\varphi \operatorname{div} U - c_8\dfrac{\varphi}{\theta} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall x \in \Omega \quad \forall t \in [0,T] \\ \theta_{|t=0} = \theta_0 \quad \varphi_{|t=0} = \varphi_0 \qquad \forall x \in \Omega \\ \theta_{|\partial\Omega} = a \quad \varphi_{|\partial\Omega} = b \qquad \forall t \in [0,T] \end{cases} \tag{2.11}$$

with $\Omega$ a bounded domain of $\mathbb{R}^N$, $\partial\Omega$ its boundary, $U$ a given function, $F$ a given non-negative function, $a$, $b$, $\nu$, $c_3$, $c_4$, $c_5$, $c_6$, $c_7$ and $c_8$ positive constants, $\theta_0 = \theta_0(x) \geq a$ and $0 < \xi \leq \varphi_0(x) \leq b$.

### 2.3.2 The reduced $\varphi$-$\theta$ model

In order to start with a simpler model, we shall modify model (2.11) by taking $\nu_t$ equal to zero and take, only for the sake of simplicity, zero Dirichelet boundary conditions.

$$\begin{cases} d_t\theta - \nu\Delta\theta = -c_3F\theta^2 + c_4\theta \operatorname{div} U + c_5 & \forall x \in \Omega \quad \forall t \in [0,T] \\ d_t\varphi - \nu\Delta\varphi = -c_6F\theta\varphi + c_7\varphi \operatorname{div} U - c_8\dfrac{\varphi}{\theta} & \forall x \in \Omega \quad \forall t \in [0,T] \\ \theta_{|t=0} = \theta_0 \quad \varphi_{|t=0} = \varphi_0 \qquad \forall x \in \Omega \\ \theta_{|\partial\Omega} = 0 \quad \varphi_{|\partial\Omega} = 0 \qquad \forall t \in [0,T] \end{cases} \tag{2.12}$$

$\theta$-equation will be modified as

$$d_t\theta - \nu\Delta\theta = -c_3F\theta|\theta| + c_4\theta \operatorname{div} U + c_5 \tag{2.13}$$

where obviously a positive solution of (2.13) is also a solution of $\theta$-equation in (2.12).

**Lemma 3 (A priori estimates)** *If $\theta_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$, if (2.9) holds and if a solution $\theta$ of (2.13) exists, then $\theta \in L^\infty(0,T; L^\infty(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$.*

**Proof:** Let us multiply (2.13) by $\theta$ and integrate over $\Omega$

$$\frac{1}{2}\partial_t\|\theta\|^2_{L^2}+\nu\|\nabla\theta\|^2_{L^2}=\int_\Omega[-c_3F|\theta|^3+c_4\theta^2\operatorname{div}U+c_5\theta]\,\mathrm{dx}-\int_\Omega U(\nabla\theta)\theta\,\mathrm{dx}. \quad (2.14)$$

Integrating by parts the last term, using Hölder's and Young's inequalities and dropping the second and the third term, we find

$$\partial_t\|\theta\|^2_{L^2}\le\left[c_5+(2c_4+1)\|\operatorname{div}U\|_{L^\infty}\right]\|\theta\|^2_{L^2}+c_5|\Omega|.$$

So, using Gronwall lemma 1,

$$\|\theta(t)\|^2_{L^2}\le\left[c_5+\|\theta_0\|^2_{L^2}\right]\exp\left[c_5|\Omega|T+(2c_4+1)\|\operatorname{div}U\|_{L^1(L^\infty)}\right]$$

and this implies that $\theta\in L^\infty(0,T;L^2(\Omega))$. In addition, integrating (2.14) in time, we have

$$\nu\|\nabla\theta\|^2_{L^2(L^2)}\le c_5\|\theta\|_{L^1(L^1)}+\frac{1}{2}\|\theta_0\|^2_{L^2}+\left(c_4+\frac{1}{2}\right)\int_0^T\!\!\int_\Omega\theta^2|\operatorname{div}U|\,\mathrm{dx}\,\mathrm{dt}$$

and it follows that $\nabla\theta\in L^2(0;T;L^2(\Omega))$ and therefore $\theta\in L^2(0,T;H^1_0(\Omega))$. Mohammadi also observes that if $\theta_0\in L^p(\Omega)$ we can multiply (2.13) by $|\theta|^{p-2}\theta$ and prove that $\theta\in L^\infty(0,T;L^p(\Omega))$ for every $p\in[1,+\infty]$.

We now want to estimate $\partial_t\theta$ multiplying (2.13) by $\partial_t\theta$ and integrating in space and time

$$\int_0^t\!\!\int_\Omega|\partial_t\theta|^2\,\mathrm{dx}\,\mathrm{dt}+\int_0^t\!\!\int_\Omega U\cdot(\nabla\theta)\partial_t\theta\,\mathrm{dx}\,\mathrm{dt}+\nu\int_0^t\!\!\int_\Omega\nabla\theta\cdot\nabla\partial_t\theta\,\mathrm{dx}\,\mathrm{dt}=$$

$$-c_3\int_0^t\!\!\int_\Omega F\theta|\theta|\partial_t\theta\,\mathrm{dx}\,\mathrm{dt}+c_4\int_0^t\!\!\int_\Omega\theta\operatorname{div}U\partial_t\theta\,\mathrm{dx}\,\mathrm{dt}+c_5\int_0^t\!\!\int_\Omega\partial_t\theta\,\mathrm{dx}\,\mathrm{dt}.$$

We note that

$$\nu\int_0^t\!\!\int_\Omega\nabla\theta\cdot\nabla\partial_t\theta\,\mathrm{dx}\,\mathrm{dt}=\frac{\nu}{2}\int_\Omega|\nabla\theta(t)|^2\,\mathrm{dx}-\frac{\nu}{2}\int_\Omega|\nabla\theta_0|^2\,\mathrm{dx},$$

$$\left|\int_0^t\!\!\int_\Omega U(\nabla\theta)\partial_t\theta\,\mathrm{dx}\,\mathrm{dt}\right|\le\|U\|_{L^\infty(L^\infty)}\|\nabla\theta\|_{L^2(L^2)}\|\partial_t\theta\|_{L^2(L^2)},$$

$$\left|\int_0^t\!\!\int_\Omega F\theta|\theta|\partial_t\theta\,\mathrm{dx}\,\mathrm{dt}\right|\le\|F\|_{L^2(L^\infty)}\|\theta\|^2_{L^\infty(L^4)}\|\partial_t\theta\|_{L^2(L^2)},$$

$$\left|\int_0^t\int_\Omega \theta \,\mathrm{div}\, U \partial_t\theta \,\mathrm{dx}\,\mathrm{dt}\right| \leq \|\partial_t\theta\|_{L^2(L^2)}\|\mathrm{div}\, U\|_{L^2(L^\infty)}\|\theta\|_{L^\infty(L^2)}$$

$$\left|\int_0^t\int_\Omega \partial_t\theta \,\mathrm{dx}\,\mathrm{dt}\right| \leq |\Omega| T\|\partial_t\theta\|_{L^2(L^2)}$$

and this means that $\partial_t\theta \in L^2(0,T;L^2(\Omega))$ and $\nabla\theta \in L^\infty(0,T;L^2(\Omega))$.

□

**Theorem 3 (Existence, uniqueness and positivity)** *If $\theta_0 \in L^\infty \cap H_0^1(\Omega)$ and if (2.9) holds then problem (2.13) has a unique not negative solution $\theta \in L^\infty(0,T;L^\infty(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$.*

**Proof:** Thanks to the a priori estimates lemma 3, the classical Faedo-Galerkin method [27] can be used to obtain the existence of $\theta$.

We now write $\theta = \theta^+ - \theta^-$, where $\theta^+ = \max(0,\theta)$ and $\theta^- = \max(0,-\theta)$ and therefore $\theta_0^- = 0$. We multiply (2.13) by $-\theta^-$ and integrate in space

$$\frac{1}{2}\partial_t\|\theta^-\|_{L^2}^2 + \nu\|\nabla\theta^-\|_{L^2}^2 =$$

$$= -c_3\int_\Omega F(\theta^-)^3\,\mathrm{dx} - \left(\frac{1}{2}+c_4\right)\int_\Omega (\theta^-)^2 \mathrm{div}\, U\,\mathrm{dx} - c_5\int_\Omega \theta^-\,\mathrm{dx}$$

and dropping the second term on the left side and the first and third term on the right side and using Gronwall lemma we get

$$\|\theta^-\|_{L^2}^2 \leq \|\theta_0^-\|_{L^2}^2 \exp\left[(1+2c_4)\|\mathrm{div}\, U\|_{L^1(L^2)}\right]$$

which means that $\theta^-$, which is zero at starting time, will always be zero and thus non-negativity is proven.

Let now suppose that two positive solutions $v_1$ and $v_2$ in $L^2(0,T;H_0^1(\Omega))$ exist and subtract their differential equations

$$\partial_t w + U\nabla w - \nu\Delta w = -c_3 F w(v_1+v_2) + c_4 w \,\mathrm{div}\, U$$

where $w = v_1 - v_2$. Multiplying by $w$ and integrating in space we obtain

$$\frac{1}{2}\partial_t\|w\|_{L^2}^2 + \nu\|\nabla w\|_{L^2}^2 = -c_3\int_\Omega F w^2(v_1+v_2)\,\mathrm{dx} + \left(c_4+\frac{1}{2}\right)\int_\Omega w^2 \mathrm{div}\, U\,\mathrm{dx}$$

and therefore, using Gronwall lemma 1 and dropping the second term on left side and the first term on right side, we get $w = 0$.

□

**Theorem 4 (Maximum principle)** *If* $\theta \in C^1([0,T];C^2(\Omega)\cap C^0(\bar{\Omega}))$, *if initial and boundary conditions are positive, then* $\theta$ *is always positive.*

**Proof:** If $\theta$ has this regularity then it is a classical solution of differential equation. We take the minimum of $\theta$ at every fixed time and suppose that there is a time when it goes to zero. Therefore at the first time that this happens

$$\partial_t\theta \geq 0 \qquad \Delta\theta \leq 0 \qquad \nabla\theta = 0$$

and differential equation in this point at this time becomes

$$0 \geq \partial_t\theta - \nu\Delta\theta = c_5 > 0,$$

which is a contradiction.

□

### The $\varphi$-equation

Under the same hypothesis on $U$ and $F$, we can prove, in the same way as we have done for $\theta$-equation, that there exist a non-negative unique solution $\varphi \in L^\infty(0,T;H_0^1(\Omega))\cap L^\infty(0,T;L^\infty(\Omega))$.

## 2.3.3 The real $\varphi$-$\theta$ system

**Theorem 5** *Assume that (2.10) holds, that* $\operatorname{div} U = 0$, *that* $\theta_0 \in L^\infty(\Omega)$ *and that* $\varphi_0 \in L^\infty(\Omega)$. *Then problem (2.11) has a unique solution in* $L^\infty(0,T;L^\infty(\Omega))$ $\cap L^2(0,T;H^1(\Omega))$ *such that* $\theta$ *is greater than* $a$ *and there is* $\lambda > 0$, *depending only on domain* $\Omega$ *and* $T$, $F$ *and* $a$, *such that* $\varphi$ *is limited from below by* $\xi e^{-\lambda T}$ *and from above by* $b$.

**Proof:** We introduce the following perturbed system

$$\begin{cases} \partial_t\theta_\epsilon + U\nabla\theta_\epsilon - \nabla(H_\epsilon(\theta_\epsilon,\varphi_\epsilon)\nabla\theta_\epsilon) = -c_3F|\theta_\epsilon|\theta_\epsilon + c_5 \\ \partial_t\varphi_\epsilon + U\nabla\varphi_\epsilon - \nabla(H_\epsilon(\theta_\epsilon,\varphi_\epsilon)\nabla\varphi_\epsilon) = -\varphi_\epsilon(c_6\theta_\epsilon F + c_8K_\epsilon(\theta_\epsilon)) \\ \theta_{\epsilon|t=0} = \theta_0 \geq a, \qquad \xi < \varphi_{\epsilon|t=0} = \varphi_0 \leq b, \\ \theta_{\epsilon|\partial\Omega} = a, \qquad \varphi_{\epsilon|\partial\Omega} = b \\ K_\epsilon(f) = (\epsilon^2+f^2)^{-\frac{1}{2}} \qquad H_\epsilon(f,g) = \nu + (\epsilon^2+f^2g^2)^{-\frac{1}{2}}. \end{cases} \tag{2.15}$$

Let $(\theta_\epsilon^n, \varphi_\epsilon^n)$ be the sequence of approximate solutions of perturbed systems (2.15) obtained with the Faedo-Galerkin method; thanks to the $L^\infty(0,T;L^\infty(\Omega) \cap L^2(0,T;H^1(\Omega))$ bounds on $(\theta_\epsilon^n, \varphi_\epsilon^n)$ and the $L^2(0,T;H^{-1}(\Omega))$ bounds on $(\partial_t\theta_\epsilon^n, \partial_t\varphi_\epsilon^n)$, $(\theta_\epsilon^n, \varphi_\epsilon^n)$ is weakly compact in $L^2(0,T;H^1(\Omega))$ and strongly compact in $L^p(0,T;L^p(\Omega))$ for any $p \geq 2$. Therefore the sequence converge to $(\theta_\epsilon, \varphi_\epsilon)$ almost everywhere as $n \to \infty$, and, thanks to the other a priori bounds, we can pass to the limit as $\epsilon \to 0$ to $(\theta, \varphi)$.

A more detailed proof can be found in [32].

□

## 2.4 Semidiscretization

Taking inspiration from the results of Mohammadi that we presented in the previous section, we present here a spatial discretization of $\varphi$-$\theta$ model (2.11) by means of finite elements and provide conditions for stability and positivity.

Let us introduce a triangulation $\tau_h$ of domain $\Omega$ composed of elements $K$.

**Definition 1** *Define $\rho_K$ as the maximum diameter of the spheres contained in the element $K$ and $h_K$ the minimum of the diameter of the spheres containing the element $K$. $h$ is the largest $h_K$.*

**Definition 2** *The triangulation $\tau_h$ is regular if and only if there is a $\sigma \geq 1$ such that for every $h > 0$ and for every element $K \in \tau_h$*

$$\frac{h_K}{\rho_K} \leq \sigma.$$

**Definition 3** *Define $X_h^k$ as the space of triangular finite elements, composed of every function in $C^0(\bar{\Omega})$ such that its restriction to element $K$ is a polynomial of degree $k$. Define $V_h = X_h^k \cap H_0^1$.*

Thanks to definition 3 for every $\theta \in H_0^1(\Omega)$ we have

$$\inf_{\theta_h \in V_h} \|\theta - \theta_h\|_{H^1(\Omega)} \to 0 \quad \text{when} \quad h \to 0.$$

Let $\theta_h$ and $\varphi_h$ be in $C^1([0,T];V_h)$ and let $\{\phi_j, j = 1, \ldots, N_h\}$ be a basis for the vector space $V_h$ so that we can set

$$\theta_h(x,t) = \sum_{j=1}^{N_h} \theta_j(t)\phi_j(x) \qquad \varphi_h(x,t) = \sum_{j=1}^{N_h} \varphi_j(t)\phi_j(x). \tag{2.16}$$

Now we want to find $\theta_h$ and $\varphi_h$ which, given initial data $\theta_{0,h} \geq \alpha > 0$ and $\varphi_{0,h} \geq \alpha > 0$, satisfy in distributional sense

$$\begin{cases} \nu_t = \dfrac{c_\mu}{\theta_h \varphi_h} \\ d_t\theta_h = \sum_{l=1}^{N} \partial_l[(\nu + c_\theta \nu_t)\partial_l \theta_h] - c_3 F \theta_h^2 + c_4 \theta_h \operatorname{div} U + c_5 \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall x \in \Omega \quad \forall t \in [0,T] \\ d_t\varphi_h = \sum_{l=1}^{N} \partial_l[(\nu + c_\varphi \nu_t)\partial_l \varphi_h] - c_6 F \theta_h \varphi_h + c_7 \varphi_h \operatorname{div} U - c_8 \dfrac{\varphi_h}{\theta_h} \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall x \in \Omega \quad \forall t \in [0,T] \\ \theta_h|_{t=0} = \theta_{0,h} \qquad \varphi_h|_{t=0} = \varphi_{0,h} \qquad \forall x \in \Omega \\ \theta_h|_{\partial\Omega} = 0 \qquad \varphi_h|_{\partial\Omega} = 0 \qquad \forall t \in [0,T] \end{cases} \tag{2.17}$$

where boundary Dirichelet conditions are taken to be zero only for sake of simplicity. If we multiply (2.17) by $\phi_i$, use (2.16) and integrate over $\Omega$, we obtain

$$\sum_{j=1}^{N_h} \partial_t \theta_j \int_\Omega \phi_j \phi_i \,\mathrm{dx} + \sum_{j=1}^{N_h} \theta_j \int_\Omega U \cdot (\nabla \phi_j) \phi_i \,\mathrm{dx} +$$

$$+ \sum_{j=1}^{N_h} \theta_j \int_\Omega \left(\nu + \frac{c_\theta c_\mu}{\theta_h \varphi_h}\right) \nabla \phi_j \cdot \nabla \phi_i \,\mathrm{dx} = -c_3 \int_\Omega F \left(\sum_{j=1}^{N_h} \theta_j \phi_j\right)^2 \phi_i \,\mathrm{dx} +$$

$$+ c_4 \sum_{j=1}^{N_h} \theta_j \int_\Omega \operatorname{div} U \phi_j \phi_i \,\mathrm{dx} + c_5 \int_\Omega \phi_i \,\mathrm{dx} \qquad \forall\ i = 1, \ldots, N_h \tag{2.18}$$

$$\sum_{j=1}^{N_h} \partial_t \varphi_j \int_\Omega \phi_j \phi_i \,\mathrm{dx} + \sum_{j=1}^{N_h} \varphi_j \int_\Omega U \cdot (\nabla \phi_j) \phi_i \,\mathrm{dx} +$$

$$+ \sum_{j=1}^{N_h} \varphi_j \int_\Omega \left(\nu + \frac{c_\varphi c_\mu}{\theta_h \varphi_h}\right) \nabla \phi_j \cdot \nabla \phi_i \,\mathrm{dx} =$$

$$= -c_6 \int_\Omega F \left(\sum_{j=1}^{N_h} \theta_j \phi_j\right) \left(\sum_{j=1}^{N_h} \varphi_j \phi_j\right) \phi_i \,\mathrm{dx} + c_7 \sum_{j=1}^{N_h} \varphi_j \int_\Omega \operatorname{div} U \phi_j \phi_i \,\mathrm{dx} +$$

$$+ c_8 \int_\Omega \frac{\sum_{j=1}^{N_h} \varphi_j \phi_j}{\sum_{j=1}^{N_h} \theta_j \phi_j} \phi_j \,\mathrm{dx} \qquad \forall\ i = 1, \ldots, N_h \tag{2.19}$$

$$\sum_{j=1}^{N_h} \theta_j(0)\phi_j = \theta_{0,h} \qquad \sum_{j=1}^{N_h} \varphi_j(0)\phi_j = \varphi_{0,h}$$

which is a system of autonomous[1] ordinary differential equations with positive initial data. Since its dependence upon $\theta_j$ and $\varphi_j$ is Lipschitz in a neighbourhood of the initial positive data, then we have a unique positive solution $\theta_h$ and $\varphi_h$ in $C^1([0,t_1);V_h)$, where $t_1$ is either $T$ or it is the time when $\theta_h$ or $\varphi_h$ goes to zero or to infinity. We are now going to show that $\theta_h$ and $\varphi_h$ are bounded and are strictly larger than a positive constant and therefore $t_1$ must be $T$.

**Theorem 6 (Positivity)** *Under the hypothesis that*

$$\operatorname{div} U \in L^\infty(0,T;L^\infty(\Omega)) \qquad F \in L^\infty(0,T;L^\infty(\Omega)) \tag{2.20}$$

*there is* $\lambda > 0$ *such that*

$$\theta_h(t) \geq \alpha \mathrm{e}^{-\lambda t} \qquad \varphi_h(t) \geq \alpha \mathrm{e}^{-\lambda t} \qquad \forall t \in [0,t_1).$$

**Proof:** Let us remind that $\alpha \leq \min(\theta_{0h}, \varphi_{0h})$ and let us introduce $\eta_h = \theta_h - \alpha \mathrm{e}^{-\lambda t}$ and, as usual, $\eta_h = \eta_h^+ - \eta_h^-$. Now $\eta_h$ satisfies in a distributional sense

$$\partial_t \eta_h + U \cdot \nabla \eta_h = \sum_{l=1}^{N} \partial_l \Big[ \Big( \nu + \frac{c_\theta c_\mu}{\theta_h \varphi_h} \Big) \partial_l \eta_h \Big] + \alpha\lambda \mathrm{e}^{-\lambda t} - c_3 F \eta_h \theta_h +$$

$$-c_3 F \theta_h \alpha \mathrm{e}^{-\lambda t} + c_4 \operatorname{div} U \eta_h + c_4 \operatorname{div} U \alpha \mathrm{e}^{-\lambda t} + c_5, \tag{2.21}$$

and multiplying (2.21) by $-\eta_h^-$ and integrating over $\Omega$ we get

$$\frac{1}{2}\partial_t \|\eta_h^-\|_{L^2}^2 \leq \frac{1}{2}\int_\Omega \operatorname{div} U (\eta_h^-)^2 \,\mathrm{dx} - \lambda \int_\Omega \alpha \mathrm{e}^{-\lambda t} \eta_h^- \,\mathrm{dx} + c_3 \int_\Omega F \theta_h \alpha \mathrm{e}^{-\lambda t} \eta_h^- \,\mathrm{dx} +$$

$$-c_3 \int_\Omega F \theta_h (\eta_h^-)^2 \,\mathrm{dx} + c_4 \int_\Omega \operatorname{div} U (\eta_h^-)^2 \,\mathrm{dx} - c_4 \int_\Omega \operatorname{div} U \alpha \mathrm{e}^{-\lambda t} \eta_h \,\mathrm{dx}$$

$$\partial_t \|\eta_h^-\|_{L^2}^2 \leq \Big[ (1+2c_4) \|\operatorname{div} U\|_{L^\infty} + 2c_3 \|F\|_{L^\infty} \|\theta_h\|_{L^\infty} \Big] \|\eta_h^-\|_{L^2}^2 +$$

$$+ \int_\Omega \eta_h^- 2\alpha \mathrm{e}^{-\lambda t} \big( -\lambda + c_3 F \theta_h - c_4 \operatorname{div} U \big) \,\mathrm{dx}.$$

[1]Autonomous means that the differential equation depends on time only through the time derivatives and the unknown; it does not explicitly depends on time. This fact has as direct consequence the fact that every solution does not depend on its starting time.

If $\lambda \geq c_3||F||_{L^\infty(L^\infty)}||\theta_h||_{L^\infty(L^\infty)}+c_4||\operatorname{div} U||_{L^\infty(L^\infty)}$, then using Gronwall lemma 1 we prove that

$$||\eta_h^-(t)||_{L^2}^2 \leq ||\eta_h^-(0)||_{L^2}^2 \cdot$$
$$\cdot \exp\Big[(1+2c_4)||\operatorname{div} U||_{L^1(L^\infty)} + 2c_3||F||_{L^2(L^\infty)}||\theta_h||_{L^2(L^\infty)}\Big].$$

An analogous inequality can be proven for $\varphi_h$ using the fact that now $\theta_h \geq \alpha e^{-\lambda t_1}$. We observe that $\theta_h$ is always in $L^\infty(0,T;L^\infty(\Omega))$ since it is a discrete function, even when the norm depends on $h$.

□

**Theorem 7 (Stability)** *If (2.20) holds, then our semidiscrete solution is in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$ with an estimate which does not depend on $h$.*

**Proof:** Let us multiply equation (2.18) by $\theta_j$ and sum over $j$

$$\frac{1}{2}\partial_t||\theta_h||_{L^2}^2 + \int_\Omega U\cdot\nabla\theta_h\,\theta_h\,\mathrm{dx} + \int_\Omega\Big(\nu + \frac{c_\theta c_\mu}{\theta_h\varphi_h}\Big)\nabla\theta_h\cdot\nabla\theta_h\,\mathrm{dx} =$$

$$= -c_3\int_\Omega F\theta_h^3\,\mathrm{dx} + c_4\int_\Omega \operatorname{div} U\theta_h^2\,\mathrm{dx} + c_5\int_\Omega\theta_h\,\mathrm{dx}$$

$$\partial_t||\theta_h||_{L^2}^2 + 2\nu||\nabla\theta_h||_{L^2}^2 \leq (2c_4+1)||\operatorname{div} U||_{L^\infty}||\theta_h||_{L^2}^2 + c_5||\theta_h||_{L^2}^2 + c_5|\Omega|$$

which proves our thesis thanks to Gronwall lemma 1.

To show our thesis for $\varphi_h$ the procedure is exactly the same, with the only difference that the last term can be dropped in the inequality.

□

**Corollary 1** *Thanks to stability theorem 7, $\theta_h$ and $\varphi_h$ cannot go to infinity in $t_1$ and therefore either $t_1 = T$ or $\theta_h$ or $\varphi_h$ goes to zero.*

**Corollary 2** *We must observe here that the constant $\lambda$ in positivity theorem 6 depends on $h$ and on $t_1$. But for every chosen $t_1$ there exists a $\lambda$ and therefore $\theta_h(t_1,x) > 0$ and $\varphi_h(t_1,x) > 0$. This means that $\varphi_h$ and $\theta_h$ cannot go to zero and therefore $t_1 = T$ and our solution is in $C^1([0,T];V_h)$.*

## 2.5 Total discretization

In this section we provide a spatial and temporal discretization of $\varphi$-$\theta$ model whose solution, under some conditions on time step $\Delta t$, is positive and stable.

We want now to find functions $\theta^{n+1}$ and $\varphi^{n+1}$ in $V_h$ such that, given $\theta^n$ and $\varphi^n$ in $V_h$, they satisfy, for every index $i = 1, \dots, N_h$

$$\sum_{j=1}^{N_h}\Big[\frac{1}{\Delta t}\int_\Omega \phi_j\phi_i \,\mathrm{dx} + \int_\Omega U(t^{n+1})\cdot\nabla\phi_j\phi_i \,\mathrm{dx} + \int_\Omega\Big(\nu + \frac{c_\theta c_\mu}{\theta^n\varphi^n}\Big)\nabla\phi_j\cdot\nabla\phi_i \,\mathrm{dx} +$$

$$+ c_3\int_\Omega F(t^{n+1})\theta^n\phi_j\phi_i \,\mathrm{dx} - c_4\int_\Omega \operatorname{div} U(t^{n+1})\phi_j\phi_i \,\mathrm{dx}\Big]\theta_j^{n+1} =$$

$$= \sum_{j=1}^{N_h}\frac{1}{\Delta t}\int_\Omega \phi_j\phi_i \,\mathrm{dx}\theta_j^n + c_5\int_\Omega \phi_i \,\mathrm{dx} \tag{2.22}$$

$$\sum_{j=1}^{N_h}\Big[\frac{1}{\Delta t}\int_\Omega \phi_j\phi_i \,\mathrm{dx} + \int_\Omega U(t^{n+1})\cdot\nabla\phi_j\phi_i \,\mathrm{dx} + \int_\Omega\Big(\nu + \frac{c_\varphi}{\theta^n\varphi^n}\Big)\nabla\phi_j\cdot\nabla\phi_i \,\mathrm{dx} +$$

$$+ c_6\int_\Omega F(t^{n+1})\theta^n\phi_j\phi_i \,\mathrm{dx} - c_7\int_\Omega \operatorname{div} U(t^{n+1})\phi_j\phi_i \,\mathrm{dx} +$$

$$+ c_8\int_\Omega \frac{1}{\theta^n}\phi_j\phi_i \,\mathrm{dx}\Big]\varphi_j^{n+1} = \sum_{j=1}^{N_h}\frac{1}{\Delta t}\int_\Omega \phi_j\phi_i \,\mathrm{dx}\varphi_j^n \tag{2.23}$$

$$\theta^0 = \theta_{0h} \qquad \varphi^0 = \varphi_{0h}.$$

Such a solution surely exists when matrices $A_\theta$ and $A_\varphi$, which multiply $\theta^{n+1}$ and $\varphi^{n+1}$ in (2.22) and (2.23), are positive definite and this happens when

$$(\Delta t)^{-1} > \Big(\frac{1}{2} + c_4\Big)\operatorname{div} U(t^{n+1}) - c_3 F(t^{n+1})\theta^n \tag{2.24}$$

$$(\Delta t)^{-1} > \Big(\frac{1}{2} + c_7\Big)\operatorname{div} U(t^{n+1}) - c_6 F(t^{n+1})\theta^n - c_8\frac{1}{\theta^n}. \tag{2.25}$$

Therefore, we can drop terms containing $\theta^n$ in (2.24) and (2.25) and simply ask

$$\Delta t < \Big[\|\operatorname{div} U\|_{L^\infty(L^\infty)}\Big(\max(c_4, c_7) + \frac{1}{2}\Big)\Big]^{-1}, \tag{2.26}$$

which is always satisfied if the fluid is incompressible.

**Theorem 8 (Positivity)** *For fixed $h$ and $\Delta t$, a solution $\theta^{n+1}$, $\varphi^{n+1}$ of (2.22) and (2.23) is always strictly positive provided that $\theta^n$ and $\varphi^n$ are strictly positive and that (2.20) and (2.26) hold.*

**Proof:** Since $\theta^n$ and $\varphi^n$ are strictly positive and they are discrete functions, then there is a $\Lambda > 0$ such that for every $\lambda \geq \Lambda$ we have[2] $\theta^n \geq \xi \mathrm{e}^{-\lambda t^n}$ and $\varphi^n \geq \xi \mathrm{e}^{-\lambda t^n}$. Let us introduce $\eta^{n+1} = \theta^{n+1} - \xi \mathrm{e}^{-\lambda t^{n+1}}$ which must satisfy, in a distributional sense,

$$\frac{1}{\Delta t}\eta^{n+1} + U(t^{n+1}) \cdot \nabla \eta^{n+1} - \nabla\left[\left(\nu + \frac{c_\theta}{\theta^n \varphi^n}\right)\nabla \eta^{n+1}\right] =$$

$$= -c_3 F(t^{n+1})\theta^n \eta^{n+1} + c_4 \mathrm{div}\, U(t^{n+1})\eta^{n+1} + c_5 + \frac{1}{\Delta t}\eta^n - \xi\lambda \mathrm{e}^{-\lambda t^s}$$

$$-c_3 F(t^{n+1})\xi \mathrm{e}^{-\lambda t^{n+1}}\theta^n + c_4 \mathrm{div}\, U(t^{n+1})\xi \mathrm{e}^{-\lambda t^{n+1}}, \qquad (2.27)$$

where $t^s$ is a time in $[t^n, t^{n+1}]$. If we split $\eta^{n+1} = (\eta^{n+1})^+ - (\eta^{n+1})^-$ and multiply (2.27) by $-(\eta^{n+1})^-$ we obtain the following inequality

$$\|(\eta^{n+1})^-\|_{L^2}^2 \leq \Delta t\left(c_4 + \frac{1}{2}\right)\int_\Omega \mathrm{div}\, U(t^{n+1})[(\eta^{n+1})^-]^2 \mathrm{dx} - \int_\Omega \eta^n (\eta^{n+1})^- \mathrm{dx} +$$

$$+\Delta t \xi \mathrm{e}^{-\lambda t^{n+1}} \int_\Omega \Big[-\lambda + c_3 F(t^{n+1})\theta^n - c_4 \mathrm{div}\, U(t^{n+1})\Big](\eta^{n+1})^- \,\mathrm{dx}$$

and if we take $\lambda$ big enough and use inequality (2.26) we get

$$\|(\eta^{n+1})^-\|_{L^2}^2 \leq 0.$$

An analogous inequality can be obtained in the same way for $\varphi^{n+1}$.

□

**Theorem 9 (Stability)** *If (2.20) holds and if $\Delta t$ is small enough*

$$(\Delta t)^{-1} > (2\max(c_4, c_7) + 1)\|\mathrm{div}\, U\|_{L^\infty(L^\infty)}, \qquad (2.28)$$

*then solution of (2.22) and (2.23) satisfies*

$$\|\theta^n\|_{L^2}^2 \leq \Big[\|\theta_{0h}\|_{L^2}^2 + Tc\Big] \exp\Big[(2c_4 + 1)\|\mathrm{div}\, U\|_{L^\infty(L^\infty)} T\Big]$$

$$\|\varphi^n\|_{L^2}^2 \leq \|\varphi_{0h}\|_{L^2}^2 \exp\Big[(2c_7 + 1)\|\mathrm{div}\, U\|_{L^\infty(L^\infty)} T\Big].$$

[2] Here $\lambda$ can depend on $h$ and on $\Delta t$.

**Proof:** We multiply equation (2.22) by $\theta_i^{n+1}$ and sum over index $i$

$$\int_\Omega (\theta^{n+1} - \theta^n)\theta^{n+1}\,\mathrm{dx} + \Delta t \int_\Omega U(t^{n+1})\cdot\nabla\theta^{n+1}\,\theta^{n+1}\,\mathrm{dx} +$$

$$+\Delta t \int_\Omega \Big(\nu + \frac{c_\theta c_\mu}{\theta^n \varphi^n}\Big)\nabla\theta^{n+1}\cdot\nabla\theta^{n+1}\,\mathrm{dx} = -c_3\Delta t\int_\Omega F(t^{n+1})(\theta^{n+1})^2\theta^n\,\mathrm{dx} +$$

$$+c_4\Delta t\int_\Omega \operatorname{div} U(t^{n+1})(\theta^{n+1})^2\,\mathrm{dx} + c_5\Delta t\int_\Omega \theta^{n+1}\,\mathrm{dx}$$

and, using

$$2\int_\Omega \Big(\theta^{n+1} - \theta^n\Big)\theta^{n+1}\,\mathrm{dx} = \|\theta^{n+1}\|_{L^2}^2 - \|\theta^n\|_{L^2}^2 + \|\theta^{n+1} - \theta^n\|_{L^2}^2 \tag{2.29}$$

we obtain the inequality

$$\|\theta^{n+1}\|_{L^2}^2 - \|\theta^n\|_{L^2}^2 + 2\nu\|\nabla\theta^{n+1}\|_{L^2}^2 \le \Delta t(2c_4+1)\|\operatorname{div} U\|_{L^\infty(L^\infty)}\|\theta^{n+1}\|_{L^2}^2 +$$

$$+\Delta t c_5\epsilon\|\theta^{n+1}\|_{L^2}^2 + \Delta t c_5\epsilon^{-1}|\Omega|,$$

where $\epsilon$ is a suitably small positive constant. Now we drop the third and fifth term and sum over index $n$ from 0 to $m-1$

$$\|\theta^m\|_{L^2}^2 - \|\theta_{0h}\|_{L^2}^2 \le \Delta t \sum_{n=1}^{m-1}(2c_4+1)\|\operatorname{div} U\|_{L^\infty(L^\infty)}\|\theta^n\|_{L^2}^2 + \tag{2.30}$$

$$+\Delta t(2c_4+1)\|\operatorname{div} U\|_{L^\infty(L^\infty)}\|\theta^m\|_{L^2}^2 +$$

$$+\Delta t(2c_4+1)\|\operatorname{div} U\|_{L^\infty(L^\infty)}\|\theta_{0h}\|_{L^2}^2 + m\Delta t c_5\epsilon^{-1}|\Omega|$$

and $\Delta t$ is small enough such that $\alpha = 1 - \Delta t(2c_4+1)\|\operatorname{div} U\|_{L^\infty(L^\infty)}$ is positive. Now we put the second and third term of the right side of (2.30) on the left side and divide by $\alpha$

$$\|\theta^m\|_{L^2}^2 - \|\theta_{0h}\|_{L^2}^2 \le$$

$$\le \frac{\Delta t}{\alpha}\sum_{n=1}^{m-1}(2c_4+1)\|\operatorname{div} U\|_{L^\infty(L^\infty)}\|\theta^n\|_{L^2}^2 + \frac{T}{\alpha}c_5\epsilon^{-1}|\Omega|.$$

Now using Gronwall discrete lemma 2 we have the thesis.

An analogous proof can be built in the same way for $\varphi^m$.

□

# Chapter 3

# Space filtered turbulence model

As it was shown in chapter 1, space filtered LES model averages Navier-Stokes equations over space using a Gaussian spatial filter and then approximates nonlinear terms using a Taylor expansion with respect to the filter width. The resulting equations for averaged quantities are like Navier-Stokes ones but with a nonlinear second-order term.

In Section 1 a global existence theorem for the solution of SF model is given provided initial data and forces are small enough. We use a standard fixed point technique (Schauder's theorem) for nonlinear problems, introducing a continuous map from a convex compact set into itself. In order to deal with the high order nonlinear term introduced by SF model, we have to use norms on Sobolev spaces of high order and accept initial data only with a small enough $H^3(\Omega)$-norm. In this way we can show that the new function is still in the starting compact set. The solution found is then in $C^0([0,T]; H^3(\Omega)) \cap L^2(0,T; H^4(\Omega))$, with $\Omega$ an open, connected and bounded subset of $\mathbb{R}^3$ or $\mathbb{R}^2$ with regular boundary.

In Section 2 we provide a uniqueness theorem for small solutions and show the existence of periodic asymptotically stable solutions and therefore the existence of stationary solutions for external forces independent of time.

**Notation:** In this chapter we will assume:

- every repeated index on the same side of an equality or inequality is summed;
- density $\rho$ is equal to one;
- the expression $DUD^2U$ means $\partial_j(\frac{\lambda^2}{2\gamma}\partial_l U_j \partial_l U_i) = \frac{\lambda^2}{2\gamma}\partial_l U_j \partial_l \partial_j U_i$. Every

calculation will be made on the former expression only for simplicity of notation;

- $c$ is an appropriate positive constant depending only on the domain $\Omega$;
- $f$ will be splitted as a sum of a gradient, which goes into the pressure term, and a divergence free field tangential to the boundary, which will still be indicated with $f$;
- $L^2(\Omega)$ is the Hilbert space of measurable functions whose square has a finite integral over $\Omega$;
- $H^m(\Omega)$ with $m \geq 1$, $m$ an integer, denotes the Hilbert space of functions in $L^2(\Omega)$ with distributional derivatives up to the $m$-th order in $L^2(\Omega)$ and is called Sobolev space;
- $H^{\frac{1}{2}}(\partial\Omega)$ is the space of the traces on the boundary of functions in $H^1(\Omega)$. $H_0^1(\Omega)$ is the space of functions in $H^1(\Omega)$ with null trace on the boundary. When $H^m(\Omega)$ with $m \geq 1$ is referred to fluid velocity, it is always $H^m(\Omega) \cap H_0^1(\Omega)$;
- a Sobolev space with index div means that its elements are divergence free;
- $\partial_t U_0$ means

$$\partial_t U_0 := -U_0 \nabla U_0 + \nu \Delta U_0 - DU_0 D^2 U_0 + f_{|t=0} - \nabla P_0. \tag{3.1}$$

Since initial value of pressure $P_0$ is not known, we have to get it applying operator $\nabla$ to differential equation (3.1)

$$\Delta P_0 = -\partial_i U_{j0} \partial_j U_{i0} - \frac{\lambda^2}{2\gamma} \partial_j \partial_l U_{i0} \partial_l \partial_i U_{j0} \quad \text{in } \Omega, \tag{3.2}$$

where boundary conditions are obtained multiplying equation (3.1) by the normal to the boundary $n$

$$\partial_n P_0 = \nu \Delta U_0 \cdot n - DU_0 D^2 U_0 \cdot n \quad \text{on } \partial\Omega.$$

In order to assure existence of $P_0$ we have to prove that

$$\int_\Omega -\partial_i U_{j0} \partial_j U_{i0} - \frac{\lambda^2}{2\gamma} \partial_j \partial_l U_{i0} \partial_l \partial_i U_{j0} \, \mathrm{dx} = \int_{\partial\Omega} \nu \Delta U_0 \cdot n - DU_0 D^2 U_0 \cdot n \, \mathrm{dx}. \tag{3.3}$$

Adding the integral over $\Omega$ of $\nu \operatorname{div} \Delta U_0$, which is zero, to the left side of (3.3) and integrating the whole side by parts, using the conditions (1.45) on initial datum, we easily get the right side.

## 3.1 Global Existence Theorem

In this section we are going to show that under compatibility conditions on initial and boundary data, if the initial values of velocity and external forces are small and regular enough, solution of SF model (1.44) exists in $C^0([0,T];H^3(\Omega)) \cap L^2(0,T;H^4(\Omega))$ for every $T > 0$.

By using a typical technique for nonlinear problems the theorem will be proved showing the existence of a fixed point for a suitable continuous map. Namely, we build a continuous map which, starting from an initial function $w$, gives another function $u$, in the same convex and compact set and such that, if $U = W$, we have found the solution of our problem. In order to have every $U$ in the same starting set as $W$, we will have to reduce the appropriate norm of initial datum and external forces. Since our fixed point map uses a differential equation to find the new $u$, we will prove that a solution exists and is unique using well–known results about linear nonstationary Navier–Stokes problem. Once divergence free velocity is found, pressure is recovered by means of orthogonality results.

**Definition 4 (Compatibility conditions)** *We say that the initial datum $U_0 \in H^3(\Omega)$ satisfies compatibility conditions if $U_0$ and $\partial_t U_0$ have null trace on the boundary of $\Omega$ and* $\operatorname{div} U_0 = 0$.

These conditions are used to assure that SF model is satisfied at initial time too in order to estimate the $L^2(H^2)$–norm in terms of the $H^1(L^2)$–norm of the solution.

**Theorem 10 (Existence)** *There exists a $\delta_0 \in \mathbb{R}^+$ such that for every $\delta \in (0, \delta_0]$ if we assume that*

1. *$\Omega$ is an open bounded connected set of $\mathbb{R}^3$ with regular boundary;*
2. *initial condition $u_0$ satisfies compatibility conditions;*
3. $\|U_0\|_{H^3} \leq \delta^2$;

*4.* $\|f\|_{L^2(H^2)} \leq \delta^2$ *and* $\|\partial_t f\|_{L^2(L^2)} \leq \delta^2$*;*

*then the solution of SF (1.44) exists in* $C^0([0,T];H^3(\Omega)) \cap L^2(0,T;H^4(\Omega))$ *for fluid velocity and* $C^0([0,T];H^2(\Omega)) \cap L^2(0,T;H^3(\Omega))$ *for pressure.*

The proof requires several steps. Let us start by defining the following set

$$A = \Big\{W : [0,T] \times \bar{\Omega} \to \mathbb{R}^3 \;\mid\; W_{|t=0} = U_0$$

$$\|W\|_{L^\infty(H^3)\cap L^2(H^4)} \leq \delta \,\wedge\, \|\partial_t W\|_{L^\infty(H^1)\cap L^2(H^2)} \leq \delta \,\wedge\, \|\partial_t \partial_t W\|_{L^2(L^2)} \leq \delta\Big\},$$

with $\delta$ positive constant which will be defined later, and build the following map, which from $W$ gives, after solving a differential problem, $U$:

$$\begin{cases} \partial_t U + \nabla P - \nu \Delta U = -W\nabla W - DWD^2W + f \\ \operatorname{div} U = 0 \\ U_{|\partial\Omega} = 0 \qquad U_{|t=0} = U_0. \end{cases} \tag{3.4}$$

We define $F =: -W\nabla W - DWD^2W + f$.

Let's introduce the orthogonal projection

$$\mathrm{Pr} : L^2(\Omega) \to L^2_{\mathrm{div}}(\Omega) = \{U \in L^2 \mid U \cdot n = 0 \,\wedge\, \operatorname{div} U = 0\};$$

We want to find $U \in L^2([0,T], H^1_{\mathrm{div}}(\Omega))$ such that

$$\begin{cases} \partial_t U - \nu \mathrm{Pr} \Delta u = -\mathrm{Pr}\Big[W\nabla W + DWD^2W\Big] + f := F(t) \\ U_{|t=0} = U_0. \end{cases} \tag{3.5}$$

Due to the orthogonal decomposition $L^2(\Omega) = L^2_{\mathrm{div}}(\Omega) \oplus G$, where $G = \{V \in L^2(\Omega) \mid V = \nabla q \,\wedge\, q \in H^1(\Omega)\}$, this problem is equivalent to (3.4).

We need, at first, the following results:

**Lemma 4** *If hypothesis of existence theorem are satisfied, then* $\|F\|_{H^1(L^2)} \leq c\delta^2$*.*

**Proof:** We start with the $L^2(L^2)$ norm of $F$ and use the fact that, since Pr is a projection, for every function $\phi$ one has $\|\mathrm{Pr}\phi\|_{L^2} \leq \|\phi\|_{L^2}$.

$$\int_0^T \|F(t)\|^2_{L^2}\,\mathrm{dt} \leq \int_0^T \|W\nabla W\|^2_{L^2}\,\mathrm{dt} + \int_0^T \|DWD^2W\|^2_{L^2}\,\mathrm{dt} + \int_0^T \|f\|^2_{L^2}\,\mathrm{dt} \leq$$

$$\leq c\|W\|^2_{L^2(H^2)}\|W\|^2_{L^\infty(H^1)} + c\|W\|^2_{L^2(H^3)}\|W\|^2_{L^\infty(H^2)} + \|f\|^2_{L^2(L^2)} \leq c\delta^4.$$

Moreover, the time derivative $\partial_t F$ satisfies

$$\int_0^T \|\partial_t F(t)\|^2_{L^2}\,\mathrm{dt} \leq$$

$$\leq \int_0^T \|\partial_t(W\nabla W)\|^2_{L^2}\,\mathrm{dt} + \int_0^T \|\partial_t(DWD^2W)\|^2_{L^2}\,\mathrm{dt} + \int_0^T \|\partial_t f\|^2_{L^2}\,\mathrm{dt} \leq$$

$$\leq c\|\partial_t W\|^2_{L^\infty(L^2)}\|W\|^2_{L^2(H^3)} + c\|W\|^2_{L^\infty(H^3)}\|\partial_t W\|^2_{L^2(H^2)} +$$

$$+\|\partial_t f\|^2_{L^2(L^2)} \leq c\delta^4.$$

□

**Lemma 5** *If hypothesis of existence theorem hold, then* $\|\partial_t U_0\|_{H^1} \leq c\delta^2$.

**Proof:** The true meaning of $\partial_t U_0$ is obviously the one given under Notation, therefore, from (3.2) we have

$$\|\nabla P_0\|_{H^1} \leq c\|\partial_i U_{j0}\partial_j U_{i0}\|_{L^2} + c\|D^2U_0D^2U_0\|_{L^2} + \|\tilde{g}\|_{H^1} \leq$$

$$\leq c\|U_0\|^2_{H^2} + c\|U_0\|^2_{H^3} + c\|U_0\|_{H^3} \leq c\delta^2,$$

where $\tilde{g} = \nu\Delta U_0 \cdot \tilde{n} - DU_0D^2U_0 \cdot \tilde{n}$, i.e. the boundary value of Neumann problem extended to the whole domain $\Omega$ with $\tilde{n}$ the normal to the boundary extended to $\Omega$.

Finally

$$\partial_t U_0 = -U_0\nabla U_0 + \nu\Delta U_0 - \nabla P_0 - DU_0D^2U_0 + f_{|t=0}$$

$$\|\partial_t U_0\|_{H^1} \leq c\|U_0\|^2_{H^2} + c\|U_0\|_{H^3} + c\|\nabla P_0\|_{H^1} + c\|U_0\|^2_{H^3} + c\|f_{|t=0}\|_{H^1} \leq c\delta^2.$$

□

We are now in a position to prove:

**Proposition 1** *Under hypothesis of existence theorem, a solution of problem (3.5) exists in* $C^0([0,T];H^3(\Omega)) \cap L^2(0,T;H^4(\Omega))$ *and*

$$\|\partial_t U\|_{L^\infty(H^1)} + \|\partial_t\partial_t U\|_{L^2(L^2)} \leq c\delta^2.$$

**Proof:** Since problem (3.4) and (3.5) are equivalent, using a result in [24, Chapter 4, Corollary 2], if $F \in H^1(0,T;L^2(\Omega))$, if compatibility conditions hold and if $\partial_t U_0 \in H^1(\Omega)$ then $U$ exists and is unique in $C^0([0,T];H^3(\Omega)) \cap L^2(0,T;H^4(\Omega))$.

Now, deriving equation (3.4) with respect to time, multiplying it by $\partial_t\partial_t U$ and integrating over $\Omega$, we get

$$\|\partial_t\partial_t U\|_{L^2}^2 - \nu\int_\Omega P\Delta\partial_t U \cdot \partial_t\partial_t U \,\mathrm{dx} = \int_\Omega \partial_t F \cdot \partial_t\partial_t U \,\mathrm{dx}$$

$$\|\partial_t\partial_t U\|_{L^2}^2 + \frac{\nu}{2}\partial_t\|\partial_t\nabla U\|_{L^2}^2 \le \epsilon\|\partial_t\partial_t U\|_{L^2}^2 + \frac{1}{4\epsilon}\|\partial_t F\|_{L^2}^2$$

Therefore, using lemma 4, taking $\epsilon = 1/2$ and integrating over $[0,T]$,

$$\|\partial_t\partial_t U\|_{L^2(L^2)}^2 + c\nu\|\partial_t U\|_{L^\infty(H^1)}^2 \le c\nu\|\partial_t U_0\|_{H^1}^2 + \|\partial_t F\|_{L^2(L^2)}^2. \tag{3.6}$$

Using compatibility conditions and thanks to lemma 5 we can state that

$$\|\partial_t U\|_{L^\infty(H^1)} + \|\partial_t\partial_t U\|_{L^2(L^2)} \le c\delta^2.$$

□

**Proposition 2** *If $\delta$ is small enough the set $A$ is not empty.*

**Proof:** We take the $H^{\frac{1}{2}}(\partial\Omega)$ function

$$\psi := \left(f_{|t=0} - DU_0 D^2 U_0 - U_0\nabla U_0 - \nabla P_0\right)_{|\partial\Omega}$$

and extend it to a function $\Psi$ in $C^0([0,T];H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, namely this function must satisfy $(\Psi_{|t=0})_{|\partial\Omega} = \psi$. This can be done thanks to [26, Volume 2, Chapter 4, Remark 3.3] (with $j = 0$, $m = 1$, $X = H^2$ and $Y = L^2$), which, in our case, states that the map which extends $H^1(\Omega)$ functions to $L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ is surjective. We then consider the following heat problem

$$\begin{cases} \partial_t V - \nu\Delta V = \Psi \\ V_{|t=0} = U_0 \quad V_{|\partial\Omega} = 0 \end{cases}$$

and observe that its solution belongs to $C^0([0,T];H^3(\Omega)) \cap L^2(0,T;H^4(\Omega))$ since $U_0 \in H^3(\Omega)$, $\Psi \in C^0([0,T];H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ and compatibility conditions (those for the heat equation, not those in Definition 4) for $V_{|t=0}$ and $(\partial_t V)_{|t=0} = \Psi_{|t=0} + \nu\Delta U_0$ are automatically satisfied from our choice of $\Psi$. Moreover, due to the bound on $U_0$ and therefore on $\Psi$, choosing $\delta$ small enough, we have that $V \in A$.

□

**Proposition 3** *The function $U$ obtained from (3.4) belongs to $A$.*

**Proof:** Since we have proven that a solution $U$ for problem (3.5) exists and that problem (3.5) is equivalent to (3.4), there exists a solution $U$,$\nabla P$ of (3.4). Observing that $\partial_t U - \nu\Delta U - F \in L^2(0,T;H^{-1}(\Omega))$ implies $\nabla P \in L^2(0,T;H^{-1}(\Omega))$, this solution satisfies the Stokes problem

$$\begin{cases} \nu\Delta U - \nabla P = \partial_t U + W\nabla W + DWD^2W - f \\ \operatorname{div} U = 0 \\ U_{|\partial\Omega} = 0. \end{cases} \tag{3.7}$$

If the term on the right side of (3.7) is in $H^2(\Omega)$, we have [24, Page 40]

$$||U||_{H^3} + ||\nabla P||_{H^1} \leq c||\partial_t U + W\nabla W + DWD^2W - f||_{H^1} \tag{3.8}$$

$$||U||_{H^4} + ||\nabla P||_{H^2} \leq c||\partial_t U + W\nabla W + DWD^2W - f||_{H^2}, \tag{3.9}$$

which implies

$$||U||_{L^\infty(H^3)} + ||\nabla P||_{L^\infty(H^1)} \leq c||\partial_t U||_{L^\infty(H^1)} + c||W||^2_{L^\infty(H^2)} +$$

$$+ c||W||^2_{L^\infty(H^3)} + c||f||_{L^\infty(H^1)} \leq c\delta^2.$$

To have the same estimate on norm $L^2(H^4)$, $U$ and $P$ can be seen as a solution of Stokes problem

$$\begin{cases} \nu\Delta\partial_t U - \nabla\partial_t P = \partial_t\partial_t U + \partial_t\big(W\nabla W\big) + \partial_t\big(DWD^2W\big) - \partial_t f \\ \operatorname{div} \partial_t U = 0 \\ \partial_t U_{|\partial\Omega} = 0. \end{cases}$$

Therefore we have

$$||\partial_t U||_{H^2} + ||\partial_t \nabla P||_{L^2} \leq c\Big\|\partial_t\partial_t U + \partial_t(W\nabla W) + \partial_t(DWD^2W) - \partial_t f\Big\|_{L^2};$$

integrating this expression and (3.9) on $[0,T]$ we get

$$||\partial_t U||_{L^2(H^2)} \leq c||\partial_t\partial_t U||_{L^2(L^2)} + c||W||_{L^2(H^4)}||\partial_t W||_{L^\infty(H^1)} +$$

$$+ c||W||_{L^\infty(H^3)}||\partial_t W||_{L^2(H^2)} + c||\partial_t f||_{L^2(L^2)} \leq c\delta^2.$$

Therefore

$$||U||_{L^2(H^4)} + ||\nabla P||_{L^2(H^2)} \leq c||\partial_t U||_{L^2(H^2)} + c||W||_{L^\infty(H^2)}||W||_{L^2(H^3)} +$$
$$+ c||W||_{L^\infty(H^3)}||W||_{L^2(H^4)} + c||f||_{L^2(H^2)} \leq c\delta^2.$$

Finally, if $\delta$ is small enough, we have shown that

$$\nabla P \in C^0([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)),$$
$$||U||_{L^\infty(H^3)} \leq \delta, \quad ||U||_{L^2(H^4)} \leq \delta,$$
$$||\partial_t U||_{L^\infty(H^1)} \leq \delta, \quad ||\partial_t U||_{L^2(H^2)} \leq \delta,$$
$$||\partial_t \partial_t U||_{L^2(L^2)} \leq \delta.$$

This means that codomain of map (3.4) is $A$.

□

**Proposition 4 (Compactness)** *$A$ is compact in $C^0([0,T]; H^2(\Omega))$.*

**Proof:** We observe that from inclusion $L^2(H^4) \cap H^1(H^2) \subset C^0(H^3)$ we can state that $A \subset C^0([0,T]; H^2(\Omega))$. From Ascoli–Arzelà theorem [13, Vol. 1, page 142] $A$ is relatively compact if and only if $A$ is equicontinuous and for every $t \in [0,T]$ the set $A(t) = \{f(t) \in H^2(\Omega) \mid \forall f \in A\}$ is relatively compact in $H^2(\Omega)$.

Since in one dimension $H^1$ functions are Hölder functions, then $A$ is immediately equicontinuous [13, Vol. 1, page 142]. Finally $A(t)$ is bounded in Hilbert space $H^3(\Omega)$ and therefore it is relatively weakly compact. Therefore, from $\{f_k\} \in A(t)$ we can extract $\{f_{k_h}\}$ which converges weakly in $H^3$ to $f$. From Rellich theorem $H^3$ is compact in $H^2$ and therefore $\{f_{k_h}\}$ converges strongly in $H^2$ to $f$. This means that $A$ is relatively compact in $C^0([0,T]; H^2)$.

To prove that $A$ is closed, we take a sequence $\phi_n \in A$ which converges in $C^0([0,T]; H^1(\Omega))$ to $\phi$. We are going to use the facts that a bounded sequence in a Hilbert space has a subsequence which converges weakly and that a bounded sequence in $L^\infty(X)$, with $X$ an Hilbert space, has a subsequence which converges weakly star; in both cases the norm of the limit function is not greater than the bound on the elements of the succession. Since $\partial_t \phi_n$ converges to $\partial_t \phi$ and $\partial_t \partial_t \phi_n$ to $\partial_t \partial_t \phi$, in the same way we have the bound on time derivatives of $\phi$. Therefore $\phi \in A$.

□

**Proposition 5 (Continuity of the map)** *Under the same hypothesis of existence theorem, map (3.4) is continuous in $C^0([0,T],H^2(\Omega))$.*

**Proof:** We take a sequence $W^k$ which converges to $W$ in $C^0([0,T];H^2(\Omega))$ and is bounded in $L^\infty(0,T;H^3(\Omega)) \cap L^2(0,T;H^4(\Omega))$. We want to show that the sequence $U^k$ created from $W^k$ by map (3.4) converges to $U$ in $C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and therefore, thanks to proposition 4, it converges in $C^0(H^2)$ and this means that our map is continuous.

We define $V^k = W - W^k$ and $S^k = U - U^k$ and we subtract differential equation for $U^k$ from the one for $U$. The result is

$$\partial_t S^k - \nu\Delta S^k = -W\nabla W + W^k\nabla W^k - DWD^2W + DW^kD^2w_k - \nabla P + \nabla P_k;$$

multiplying by $S^k$ and integrating over $\Omega$ we get

$$\frac{1}{2}\partial_t\|S^k\|^2_{L^2} + \nu\|S^k\|^2_{H^1} \leq$$

$$\leq \int_\Omega\left|S^kV^k\nabla W\right| \mathrm{dx} + \int_\Omega\left|S^kW^k\nabla V^k\right| \mathrm{dx} +$$

$$+ \int_\Omega\left|S^kDV^kD^2W\right| \mathrm{dx} + \int_\Omega\left|S^kDW^kD^2V^k\right| \mathrm{dx} \leq$$

$$\leq c\|S^k\|_{L^2}\|V^k\|_{H^1}\|W\|_{H^2} + c\|S^k\|_{L^2}\|W^k\|_{H^2}\|V^k\|_{H^1} +$$

$$+c\|S^k\|_{L^2}\|V^k\|_{H^2}\|W\|_{H^3} + c\|S^k\|_{L^2}\|W^k\|_{H^3}\|V^k\|_{H^2} \leq$$

$$\leq \epsilon\|S^k\|^2_{L^2} + \frac{c}{4\epsilon}\delta^2\|V^k\|^2_{H^2}.$$

Taking $\epsilon = \nu/2$ and integrating on $[0,T]$, we have

$$\|S^k\|^2_{L^2(H^1)} + \|S^k\|^2_{L^\infty(L^2)} \leq$$

$$\leq c\delta^2\|V^k\|^2_{L^2(H^2)} \to 0.$$

□

We can now conclude the proof of existence theorem 10. In proposition 5 we have proven that map (3.4) is continuous and therefore, since $A$ is a non-empty, convex and compact set in the Banach space $C^0([0,T];H^2(\Omega))$, using Schauder's theorem there exists a fixed point $u$ of map (3.4). Clearly, this fixed point is a solution $u$ of SF model, which is small in $C^0([0,T];H^3(\Omega)) \cap L^2(0,T;H^4(\Omega))$. The corresponding pressure $P$ satisfies $\nabla P \in C^0(0,T;H^1(\Omega) \cap L^2(0,T;H^2(\Omega))$.

## 3.2 Uniqueness and Periodic Solutions

**Theorem 11 (Uniqueness)** *Under the same hypothesis of existence theorem, for $\delta$ small enough a solution of SF model $U \in A$ is unique in $A$, while $\nabla P$ is unique.*

**Proof:** We define $S = U - V$, which satisfies

$$\begin{cases} \partial_t S - \nu \Delta S = -U\nabla U + V\nabla V - DUD^2U + DVD^2V - \nabla P_U + \nabla P_V \\ \operatorname{div} S = 0 \\ S_{|t=0} = 0 \quad S_{|\partial\Omega} = 0. \end{cases} \tag{3.10}$$

If we multiply by $S$, integrate over $\Omega$ and use the fact that $\operatorname{div} S = 0$ we get

$$\frac{1}{2}\partial_t \|S\|^2_{L^2} + \nu \|S\|^2_{H^1} \leq$$

$$\leq \int_\Omega \left| S_j \partial_j U_i S_i \right| \mathrm{dx} + c \int_\Omega \left( \left| \partial_l S_j \partial_j \partial_l U_i S_i \right| + \left| \partial_l V_j \partial_l S_i \partial_j S_i \right| \right) \mathrm{dx} \leq$$

$$\leq c\|S\|^2_{L^2}\|U\|_{H^3} + c\|S\|^2_{H^1}\left(\|U\|_{H^3} + \|V\|_{H^3}\right).$$

Integrating on $[0, T]$ we can obtain, remembering that $U$ and $V$ are small in $L^\infty(0, T; H^3(\Omega))$,

$$\|S(t)\|^2_{L^2} + \|S\|^2_{L^2(H^1)} \leq c\delta \|S\|^2_{L^2(H^1)},$$

$$\|S(t)\|^2_{L^2} \leq 0.$$

Uniqueness of $\nabla P$ follows from system (3.10).

□

To have uniqueness of $P$ a further condition on $P$ must be imposed, such as

$$\int_\Omega P \, \mathrm{dx} = 0.$$

**Theorem 12 (Stability)** *Let $V$ and $U$ be two solutions in $A$ with different initial values. If $\delta$ is small enough, the $L^2(\Omega)$ norm of their difference is controlled by the $L^2(\Omega)$ norm of the difference of their initial values and decreases exponentially with time.*

**Proof:** Let $S = V - U$. We have

$$\begin{cases} \partial_t S - \nu \Delta S = -V\nabla S - S\nabla U - DVD^2S - DSD^2U - \nabla(P_V - P_U) \\ \operatorname{div} S = 0 \\ S_{|t=0} = V_0 - U_0 \quad S_{|\partial\Omega} = 0. \end{cases} \tag{3.11}$$

We now multiply against $S$ and integrate over $\Omega$

$$\frac{1}{2}\partial_t \|S\|^2_{L^2} + \nu \|S\|^2_{H^1} \leq$$

$$\leq c\Big[\|S\|^2_{L^2}\|U\|_{H^3} + \|S\|^2_{H^1}\|V\|_{H^3} + \|S\|^2_{H^1}\|U\|_{H^3}\Big] \leq c\delta \|S\|^2_{H^1},$$

having integrated by parts the terms $\int_\Omega V(\nabla S)S\,\mathrm{dx}$ and $\int_\Omega DV(D^2S)S\,\mathrm{dx}$. Changing the value of $c$ we easily have for $\delta$ small enough

$$\partial_t \|S\|^2_{L^2} + c_0\|S\|^2_{L^2} \leq 0;$$

$$\frac{\partial}{\partial t}\left(\mathrm{e}^{c_0 t}\|S\|^2_{L^2}\right) \leq 0;$$

$$\|V(t) - U(t)\|^2_{L^2} = \|S(t)\|^2_{L^2} \leq \mathrm{e}^{-c_0 t}\|S(0)\|^2_{L^2} = \mathrm{e}^{-c_0 t}\|V_0 - U_0\|^2_{L^2}.$$

□

**Theorem 13 (Periodic Solution)** *Let $f$ be periodic of period $T > 0$ and let hypothesis of existence theorem be satisfied with $\delta^2$ instead of $\delta$. Then there exists a periodic solution of period $T$ which is asymptotically stable and unique among any other solution which satisfies existence theorem.*

**Proof:** If $\delta^2$ is used instead of $\delta$, our solution is smaller than $\delta^2$ and initial datum is smaller than $\delta^4$.

We will follow here the approach of Serrin [41]. Let $U$ be the solution of SF model with $U_0$ as initial value; let's define

$$\Phi_n(x) = U(nT, x) \quad \forall n \in \mathbb{N};$$

we want to show now that $\Phi_n$ is a Cauchy's sequence in $L^2(\Omega)$. Therefore we take two natural indexes, $n$ and $m$, with $m > n$ and define

$$W(t, x) = U(t + (m - n)T, x).$$

This means that $W$ is a solution of SF model with initial value $U((m-n)T,x)$. Thanks to stability theorem we get

$$\|W(t)-U(t)\|_{L^2}^2 \leq \mathrm{e}^{-c_0 t}\|U_0 - U((m-n)T,x)\|_{L^2} \leq 2\delta \mathrm{e}^{-c_0 t}$$

which, taking $t = nT$, becomes

$$\|U(mT)-U(nT)\|_{L^2}^2 = \|W(nT)-U(nT)\|_{L^2}^2 \leq 2\delta \mathrm{e}^{-c_0 nT}.$$

Therefore $\Phi_n$ is a Cauchy's sequence and $\Phi_n \to \Phi$ in $L^2(\Omega)$. The function $\Phi$ is also the weak limit in $H^3$, strong limit in $H^m$ for every $m < 3$, in particular the uniform limit of $\Phi_n$. Moreover, since the weak limit of a succession in $H_0^1$ remains in $H_0^1$, we have that compatibility conditions on $\partial_t\Phi$, which are

$$\partial_t \Phi = -\Phi\nabla\Phi + \nu\Delta\Phi - D\Phi D^2\Phi + f_{|t=0} - \nabla P_\Phi,$$

where $P_\Phi$ satisfies

$$\Delta P_\Phi = -\partial_i U_{j0}\partial_j U_{i0} - \frac{\lambda^2}{2\gamma}\partial_j\partial_l U_{i0}\partial_l\partial_i U_{j0} \quad \text{in } \Omega$$

$$\partial_n P_\Phi = \nu\Delta U_0 \cdot n - DU_0 D^2 U_0 \cdot n \quad \text{on } \partial\Omega,$$

are satisfied for every $\Phi_n$ since they are solutions of SF model calculated at different times and $f_{|t=0} = f_{|t=nT}$, and therefore they are satisfied for $\Phi$ too. For the same reason, the divergence of $\Phi$ is zero and the $H^3(\Omega)$–norm of $\Phi$ is smaller than $c\delta^2$.

We have now to show that a solution $V$ having $\Phi$ as initial value is periodic. Let's define

$$\bar{V}(t,x) = U(t+nT,x);$$

since $f$ is periodic, $\bar{V}$ is a solution with initial value $\Phi_n(x)$ and therefore

$$\|V(t)-\bar{V}(t)\|_{L^2}^2 \leq \mathrm{e}^{-c_0 t}\|\Phi - \Phi_n\|_{L^2}^2.$$

Taking $t = T$ we have

$$\|V(T)-\Phi_{n+1}\|_{L^2}^2 \leq \mathrm{e}^{-c_0 T}\|\Phi - \Phi_n\|_{L^2}^2,$$

which becomes, when $n \to \infty$,

$$V(T) = \Phi = V(0).$$

Uniqueness follows from the fact that $V$ is asymptotically stable.

□

**Corollary 3 (Stationary Solution)** *Under the same hypothesis of periodic solution theorem, if $f$ is time–independent, the asymptotically stable solution is constant in time.*

**Proof:** A constant function is periodic of period $1/n$, for every natural $n$. Therefore, once taken an initial value, $V$ is unique and periodic for every $\mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $V$ is constant.

□

# Chapter 4

# Eddy viscosity SF turbulence model

As it was shown in Chapter 1, eddy viscosity space filtered (EVSF) large eddy simulation model (4.1) averages Navier-Stokes equations over space using a Gaussian spatial filter, approximates nonlinear terms using a Taylor expansion with respect to the filter width and approximates the SGS terms using a Smagorinsky model. The resulting equations for averaged quantities are like Navier-Stokes ones but with a nonlinear second-order term and a non-constant eddy viscosity summed to the kinematic viscosity.

In Section 3 a global existence theorem for the solution of EVSF model is given without the typical hypothesis that initial data be small. This is due to the presence of eddy viscosity which manages to control the convective and the SF term. We use a standard Galerkin technique, building a priori estimates in Section 2 and then showing that the Galerkin approximation of the solution converges thanks to these estimates and to positive properties of the eddy viscosity term. In order to deal with the high order nonlinear term introduced by EVSF model, we have to use norms on Sobolev spaces of high order and accept initial data only in $H^2(\Omega)$. The solution found is then in $H^1(0,T;L^2(\Omega)) \cap L^{2+2\mu}(0,T;W^{1,2+2\mu}(\Omega))$, with $\Omega$ an open, connected and bounded subset of $\mathbb{R}^N$ with regular boundary.

In Section 4 we provide a uniqueness theorem and show that under certain hypothesis our solution decays to zero.

## 4.1 The problem

EVSF large eddy simulation model, often simply called Large Eddy Simulation, can be stated as:

$$\begin{cases} \partial_t U_i + \sum_{j=1}^{N} U_j \partial_j U_i = -\frac{1}{\rho}\partial_i P + \sum_{j=1}^{N} \partial_j \Big[(\nu + C\|\nabla U\|^{2\mu})\partial_j U_i\Big] + \\ \qquad\qquad - \sum_{j,l=1}^{N} \partial_j \Big[\frac{\lambda^2}{2\gamma}\partial_l U_i \partial_l U_j\Big] + f_i \qquad i = 1, \dots, N \\ \sum_{j=1}^{N} \partial_j U_j = 0,. \end{cases} \tag{4.1}$$

$$U_{|\partial\Omega} = 0 \qquad U_{|t=0} = U_0 \qquad \sum_{j=1}^{N} \partial_j U_{0j} = 0 \qquad U_{0|\partial\Omega} = 0,$$

where $\mu \geq 0.5$ and, in numerical computations, is usually taken equal to 0.5, $\gamma$ is usually taken equal to 6, $\lambda$ is usually taken equal to the grid size and $C$ is usually the square of the grid size. It is interesting to observe that the coefficient of the eddy viscosity term is usually taken as twelve times the coefficient of the other turbulent term.

For sake of simplicity, we changed the second term of right side of this model by substituting mean velocity deformation tensor $D$ with the mean velocity gradient $\nabla U$. We observe also that for $\lambda = 0$ the study of solution of this problem can be found in [23] without smallness conditions on initial datum while for $C = 0$ the study of solution can be found in the previous chapter and in [10] only in the case of small initial datum.

**Notation:** In this chapter we will assume:

- a norm without index is the standard vector norm of $\mathbb{R}^N$ not a functional space norm, therefore it is still a function of time and space;
- a functional space norm of a vector function is the functional space norm of the vector norm of that function;
- every repeated index on the same side of an equality or inequality is summed;
- density $\rho$ is equal to one;

- $c$ (and not $C$) is an appropriate positive constant depending only on the size of domain $\Omega$ and on the constants of our problem ($C$, $\nu$, $\mu$, $N$, $\lambda$ and $\gamma$);
- $f$ will be splitted as a sum of a gradient, which goes into the pressure term, and a divergence free field tangential to the boundary, which will still be indicated with $f$;
- $L^p(\Omega)$ is the space of measurable functions whose $p$-th power has a finite integral over $\Omega$; it is a Hilbert space when $p=2$;
- $W^{m,p}(\Omega)$ with $m \geq 1$, $m$ an integer, denotes the space of functions in $L^p(\Omega)$ with distributional derivatives up to the $m$-th order in $L^p(\Omega)$;
- $H^m(\Omega)$ denotes the Hilbert $W^{m,2}(\Omega)$ and is called Sobolev space;
- $H^{\frac{1}{2}}(\partial\Omega)$ is the space of the traces on the boundary of functions in $H^1(\Omega)$. $H_0^1(\Omega)$ is the space of functions in $H^1(\Omega)$ with null trace on the boundary. When $H^m(\Omega)$ with $m \geq 1$ is referred to fluid velocity, it is always $H^m(\Omega) \cap H_0^1(\Omega)$;
- a space with index div means that its elements are divergence free;
- $\partial_t U_0$ means

$$\partial_t U_0 := -U_{j0}\partial_j U_0 + \partial_j\Big[(\nu + C\|\nabla U_0\|^{2\mu})\partial_j U_0\Big] + \\ -\frac{\lambda^2}{2\gamma}\partial_l\partial_j U_0 \partial_l U_{j0} + f_{|t=0} - \nabla P_0. \tag{4.2}$$

Since initial value of pressure $P_0$ is not known, we have to get it applying divergence operator to differential equation (4.2)

$$\Delta P_0 = -\partial_i U_{j0}\partial_j U_{i0} + C\partial_j\Big[\partial_i\|\nabla U_0\|^{2\mu}\partial_j U_{i0}\Big] + \\ -\frac{\lambda^2}{2\gamma}\partial_l\partial_j U_{i0}\partial_l\partial_i U_{j0} \quad \text{in } \Omega, \tag{4.3}$$

where boundary conditions are obtained multiplying equation (4.2) by the normal to the boundary $n$

$$\partial_n P_0 = \partial_j\Big[\big(\nu + C\|\nabla U_0\|^{2\mu}\big)\partial_j U_0\Big]\cdot n - \frac{\lambda^2}{2\gamma}\partial_l\partial_j U_0\partial_l U_{j0}\cdot n \quad \text{on } \partial\Omega.$$

In order to assure existence of $P_0$ we have to prove that

$$\int_\Omega -\partial_i U_{j0}\partial_j U_{i0} - \frac{\lambda^2}{2\gamma}\partial_j\partial_l U_{i0}\partial_l\partial_i U_{j0} + C\partial_j\left[\partial_i\|\nabla U\|^{2\mu}\partial_j U_{i0}\right] \mathrm{dx} =$$

$$= \int_{\partial\Omega}\partial_j\left[\left(\nu + C\|\nabla U_0\|^{2\mu}\right)\partial_j U_0\right]\cdot n - \frac{\lambda^2}{2\gamma}\partial_l\partial_j U_0\partial_l U_{j0}\cdot n\,\mathrm{dx}. \qquad (4.4)$$

Integrating by parts the left side of (4.4), using the conditions (1.45) on initial datum, we easily get the right side.

## 4.2 A priori estimates

**Definition 5 (Compatibility conditions)** *We say that the initial datum $U_0 \in H^2(\Omega)$ satisfies compatibility conditions if $U_0$ and $\partial_t U_0$ have null trace on the boundary of $\Omega$ and* $\operatorname{div} U_0 = 0$.

These conditions are used to assure that EVSF model is satisfied at initial time too in order to estimate the $L^2$-norm of time derivative of initial velocity in terms of the $L^2$-norm of $\|\nabla U_0\|^{2\mu}\Delta U_0$.

We start by showing these lemmae which will be used during the proof of the following theorems.

**Lemma 6** *Let $\beta$ be a positive real number, let $a$, $b$ and $\alpha$ be three non-negative real numbers and let $p$ and $q$ be two real numbers larger than 1 such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

*Then we have*

$$\alpha a^{\frac{1}{p}} b^{\frac{1}{q}} \le \beta a + \frac{\alpha^q}{q} b(\beta p)^{-\frac{q}{p}}.$$

**Proof:** Using Young's inequality

$$\alpha a^{\frac{1}{p}} b^{\frac{1}{q}} \le \frac{\epsilon^p a}{p} + \frac{\alpha^q b}{\epsilon^q q}$$

and choosing

$$\epsilon = (\beta p)^{\frac{1}{p}}$$

we get the thesis.

□

**Lemma 7** *Let $a_{ki}$, $b_{kj}$ and $c_{ji}$ be three $N \times N$ matrices with non-negative elements. Then*

$$\sum_{i,j,k} a_{ki} b_{kj} c_{ji} \leq \|a\| \, \|b\| \, \|c\|.$$

**Proof:** To prove this inequality we simply

$$\sum_{i,j,k} a_{ki} b_{kj} c_{ji} = \sum_{i,j} \left[ \sum_{k} a_{ki} b_{kj} \right] c_{ji} \leq \left[ \sum_{i,j} \left[ \sum_{k} a_{ki} b_{kj} \right]^2 \right]^{\frac{1}{2}} \left[ \sum_{i,j} c_{ji}^2 \right]^{\frac{1}{2}} \leq$$

$$\leq \left[ \sum_{i,j} \left( \sum_{k} a_{ki}^2 \right) \left( \sum_{k} b_{kj}^2 \right) \right]^{\frac{1}{2}} \left[ \sum_{i,j} c_{ji}^2 \right]^{\frac{1}{2}} =$$

$$= \left[ \sum_{i,k} a_{ki}^2 \right]^{\frac{1}{2}} \left[ \sum_{j,k} a_{kj}^2 \right]^{\frac{1}{2}} \left[ \sum_{i,j} a_{ji}^2 \right]^{\frac{1}{2}} = \|a\| \, \|b\| \, \|c\|,$$

where we used Hölder inequality to prove the two inequalities.

□

**Lemma 8** *Let $b_{ji}$ be a $N \times N$ matrix with non-negative elements and $a_i$ be a $N$ vector with non-negative elements. Then*

$$\sum_{i,j} a_j b_{ji} a_i \leq \|a\|^2 \, \|b\|.$$

**Proof:** This lemma is proven in exactly the same way as the previous one.

□

**Theorem 14 (First a priori estimate)** *Assuming that $U_0 \in L^2(\Omega)$ and that $f \in L^2(0,T;L^2(\Omega))$, if a solution of problem (4.1) exists in a distributional sense and if*

$$C > \frac{\lambda^2}{2\gamma} \qquad \text{when } \mu = 0.5,$$

*then*

$$\|U\|^2_{L^\infty(L^2)} + \|\nabla U\|^2_{L^2(L^2)} + \|\nabla U\|^2_{L^{2+2\mu}(L^{2+2\mu})} \leq$$
$$\leq cT + c\|f\|^2_{L^2(L^2)} + c\|U_0\|^2_{L^2}$$

*with $c = c(\nu, \mu, |\Omega|, C, \lambda, \gamma)$.*

**Proof:** We multiply the first three differential equations (4.1) by $U_i$, sum over index $i$ and integrate over $\Omega$

$$\frac{1}{2}\partial_t \|U\|^2_{L^2} + \nu \|\nabla U\|^2_{L^2} + C \int_\Omega \|\nabla U\|^{2+2\mu}\,\mathrm{dx} \leq$$
$$\leq \frac{\lambda^2}{2\gamma}\int_\Omega \sum_{i,j,l} |\partial_l U_i \partial_l U_j \partial_j U_i|\,\mathrm{dx} + \|f\|_{L^2}\|U\|_{L^2} \leq$$
$$\leq \frac{\lambda^2}{2\gamma}\int_\Omega \|\nabla U\|^3\,\mathrm{dx} + \|f\|_{L^2}\|U\|_{L^2},$$

where lemma 7 has been used. The right side term can be estimated when $\mu = 0.5$ with

$$\frac{\lambda^2}{2\gamma}\|\nabla U\|^{2+2\mu}_{L^{2+2\mu}} + \frac{c}{2\nu}\|f\|^2_{L^2} + \frac{\nu}{2}\|\nabla U\|^2_{L^2}$$

and integrating in time we easily get the thesis. If $\mu > 0.5$ we estimate it with

$$\frac{\lambda^2}{2\gamma}\|\nabla U\|^3_{L^{2+2\mu}}|\Omega|^{1-\frac{3}{2+2\mu}} + \frac{\nu}{2c}\|U\|^2_{L^2} + \frac{c}{2\nu}\|f\|^2_{L^2} \leq$$
$$\leq \epsilon \|\nabla U\|^{2+2\mu}_{L^{2+2\mu}} + \frac{c}{\epsilon} + \frac{\nu}{2}\|\nabla U\|^2_{L^2} + \frac{c}{2\nu}\|f\|^2_{L^2}.$$

Taking $\epsilon$ small enough and integrating in time we get the thesis.

□

**Theorem 15 (Second a priori estimate)** *Assuming that $U_0 \in L^2(\Omega)$, that $f \in L^2(0,T;L^2(\Omega))$ and that $\partial_t f \in L^2(0,T;L^2(\Omega))$, if a solution of problem (4.1) exists in a distributional sense and if*

$$\begin{cases} C > \dfrac{\lambda^2}{\gamma} & \textit{when } \mu = 0.5 \\ C > \left(\dfrac{\lambda^2}{\gamma}\right)^{2\mu} \dfrac{1}{2\mu}\left[\dfrac{4\mu - 2}{\nu\mu}\right]^{2\mu-1} & \textit{when } \mu > 0.5. \end{cases} \tag{4.5}$$

*then*

$$\|\partial_t U\|^2_{L^\infty(L^2)} + \|\nabla U\|^2_{L^\infty(L^2)} + \|\nabla U\|^{2+2\mu}_{L^\infty(L^{2+2\mu})} + \|\nabla \partial_t U\|^2_{L^2(L^2)} +$$
$$+ \int_0^T\!\!\int_\Omega \|\nabla\partial_t U\|^2 \|\nabla U\|^{2\mu}\,\mathrm{dx}\,\mathrm{dt} + \int_0^T\!\!\int_\Omega (\nabla U\cdot\partial_t \nabla U)^2 \|\nabla U\|^{2\mu-2}\,\mathrm{dx}\,\mathrm{dt} \leq$$
$$\leq c\Big[\|\partial_t U_0\|^2_{L^2} + \|\nabla U_0\|^2_{L^2} + \|\nabla U_0\|^{2+2\mu}_{L^{2+2\mu}} + \|f\|^2_{L^2(L^2)} + \|\partial_t f\|^2_{L^2(L^2)}\Big]\mathrm{e}^{cT}.$$

**Proof:**

We derive in time the first three differential equations (4.1)

$$\partial_t\partial_t U_i + \partial_t U_j \partial_j U_i + U_j \partial_j \partial_t U_i - \nu \partial_t \Delta U_i - C\partial_j \Big[\partial_j \partial_t U_i \|\nabla U\|^{2\mu}\Big] +$$

$$-2\mu C \partial_j \Big[\partial_j U_i \|\nabla U\|^{2\mu-2} \partial_h U_k \partial_t \partial_h U_k\Big] =$$

$$= -\partial_i \partial_t P - \frac{\lambda^2}{2\gamma}\partial_j[\partial_l \partial_t U_i \partial_l U_j] - \frac{\lambda^2}{2\gamma}\partial_j[\partial_l U_i \partial_l \partial_t U_j] + \partial_t f_i$$

and multiply by $\partial_t U_i$, sum over index $i$ and integrate over $\Omega$

$$\frac{1}{2}\partial_t \|\partial_t U\|^2_{L^2} + \int_\Omega \partial_t U_j \partial_j U_i \partial_t U_i \,\mathrm{dx} + C\int_\Omega \|\partial_t \nabla U\|^2 \|\nabla U\|^{2\mu}\,\mathrm{dx} +$$

$$+\nu \|\partial_t \nabla U\|^2_{L^2} + 2\mu C\int_\Omega \partial_j U_i \partial_j \partial_t U_i \|\nabla U\|^{2\mu-2}\partial_h U_k \partial_t \partial_h U_k \,\mathrm{dx} =$$

$$= \frac{\lambda^2}{2\gamma}\int_\Omega \Big(\partial_l \partial_t U_i \partial_l U_j \partial_j \partial_t U_i + \partial_l U_i \partial_t \partial_l U_j \partial_j \partial_t U_i\Big)\,\mathrm{dx} + \int_\Omega \partial_t f_i \partial_t U_i \,\mathrm{dx} \leq$$

$$\leq \frac{\lambda^2}{\gamma}\int_\Omega \|\partial_t \nabla U\|^2 \|\nabla U\|\,\mathrm{dx} + \int_\Omega \partial_t f_i \partial_t U_i \,\mathrm{dx} \tag{4.6}$$

We are now going to estimate the first term of the right side. Since we will need exact estimate's constants, we will go into deep details. In the case $\mu > 0.5$ we have

$$\frac{\lambda^2}{\gamma}\int_\Omega \|\partial_t \nabla U\|^2 |\nabla U\|\,\mathrm{dx} = \frac{\lambda^2}{\gamma}\int_\Omega \|\partial_t \nabla U\|^{\frac{1}{\mu}} \|\nabla U\|\, \|\partial_t \nabla U\|^{\frac{2\mu-1}{\mu}}\,\mathrm{dx} \leq$$

$$\leq \frac{\lambda^2}{\gamma}\left[\int_\Omega \|\partial_t \nabla U\|^2 \|\nabla U\|^{2\mu}\,\mathrm{dx}\right]^{\frac{1}{2\mu}} \left[\int_\Omega \|\partial_t \nabla U\|^2\,\mathrm{dx}\right]^{\frac{2\mu-1}{2\mu}} \leq$$

$$\leq \frac{\nu}{4}\|\partial_t \nabla U\|^2_{L^2} + \left(\frac{\lambda^2}{\gamma}\right)^{2\mu} \frac{1}{2\mu}\left[\frac{4\mu-2}{\mu\nu}\right]^{2\mu-1} \int_\Omega \|\nabla U\|^{2\mu} \|\partial_t \nabla U\|^2\,\mathrm{dx} \tag{4.7}$$

and in the case $\mu = 0.5$ we do not need any other estimate.

We now take the first three differential equations (4.1) and multiply them by $\partial_t U_i$, sum over $i$ and integrate over $\Omega$

$$\|\partial_t U\|^2_{L^2} + \int_\Omega U_j \partial_j U_i \partial_t U_i \,\mathrm{dx} + \frac{\nu}{2}\partial_t \|\nabla U\|^2_{L^2} + C\int_\Omega \nabla U \cdot \partial_t \nabla U \|\nabla U\|^{2\mu}\,\mathrm{dx} =$$

$$= \frac{\lambda^2}{2\gamma}\int_\Omega \partial_l U_i \partial_l U_j \partial_j \partial_t U_i \, dx + \int_\Omega f_i \partial_t U_i \, dx \tag{4.8}$$

Let us estimate the first term of the right side whose absolute value, when $\mu < 2$, is smaller than

$$\frac{\lambda^2}{2\gamma}\int_\Omega \|\nabla U\|^2 \|\partial_t \nabla U\| \, dx = \frac{\lambda^2}{2\gamma}\int_\Omega \|\nabla U\|^\mu \|\partial_t \nabla U\| \, \|\nabla U\|^{2-\mu} \, dx \leq$$

$$\leq \frac{\lambda^2}{2\gamma}\left[\int_\Omega \|\nabla U\|^{2\mu}\|\partial_t \nabla U\|^2 \, dx\right]^{\frac{1}{2}}\left[\int_\Omega \|\nabla U\|^{4-2\mu} \, dx\right]^{\frac{1}{2}} \leq$$

$$\leq C\epsilon \int_\Omega \|\nabla U\|^{2\mu}\|\partial_t \nabla U\|^2 \, dx + c(\lambda,\gamma,C,\mu,\epsilon)\int_\Omega \|\nabla U\|^{2+2\mu} \, dx + c(\mu)|\Omega| \tag{4.9}$$

When $\mu = 2$ we have

$$\frac{\lambda^2}{2\gamma}\int_\Omega \|\nabla U\|^2\|\partial_t \nabla U\| \, dx \leq C\epsilon \int_\Omega \|\nabla U\|^4 \|\partial_t \nabla U\|^2 \, dx + c(\lambda,\gamma,C,\epsilon,|\Omega|), \tag{4.10}$$

while when $\mu > 2$ we get

$$\frac{\lambda^2}{2\gamma}\int_\Omega \|\nabla U\|^2\|\partial_t \nabla U\| \, dx = \frac{\lambda^2}{2\gamma}\int_\Omega \|\nabla U\|^2 \|\partial_t \nabla U\|^{\frac{2}{\mu}} \|\partial_t \nabla U\|^{\frac{\mu-2}{\mu}} \, dx \leq$$

$$\leq \frac{\lambda^2}{2\gamma}\left[\int_\Omega \|\nabla U\|^{2\mu}\|\partial_t \nabla U\|^2 \, dx\right]^{\frac{1}{\mu}}\left[\int_\Omega \|\partial_t \nabla U\|^{\frac{\mu-2}{\mu-1}} \, dx\right]^{\frac{\mu-1}{\mu}} \leq$$

$$\leq \frac{\nu}{8}\int_\Omega \|\partial_t \nabla U\|^2 \, dx + c(\lambda,\mu,\epsilon,\nu,C,|\Omega|) + C\epsilon\int_\Omega \|\nabla U\|^{2\mu}\|\partial_t \nabla U\|^2 \, dx. \tag{4.11}$$

Let us evaluate the last term of the left side of (4.8)

$$C\int_\Omega \Big(\sum_{h,k} \partial_h U_k \partial_t \partial_h U_k\Big)\Big(\sum_{h,k}(\partial_h U_k)^2\Big)^\mu dx =$$

$$= \frac{C}{2\mu+2}\frac{\partial}{\partial t}\int_\Omega \Big(\sum_{h,k}(\partial_h U_k)^2\Big)^{\mu+1} dx = \frac{C}{2\mu+2}\frac{\partial}{\partial t}\int_\Omega \|\nabla U\|^{2+2\mu} \, dx. \tag{4.12}$$

The last thing we must now estimate are convective terms of equations (4.6) and (4.8). The absolute value of the first is estimated, using lemma 8, Hölder inequality and embedding theorem, as

$$\sum_{i,j}\int_\Omega |\partial_t U_j| \, |\partial_j U_i| \, |\partial_t U_i| \, dx \leq \int_\Omega \|\partial_t U\|^2 \|\nabla U\| \, dx \leq$$

$$\leq \left[\int_\Omega \|\partial_t U\|^6 \,\mathrm{dx}\right]^{\frac{1}{6}} \left[\int_\Omega \|\partial_t U\|^{\frac{6}{5}} \|\nabla U\|^{\frac{6}{5}} \,\mathrm{dx}\right]^{\frac{5}{6}} \leq$$

$$\leq \frac{\nu}{16}\|\partial_t \nabla U\|^2_{L^2} + c\left[\int_\Omega \|\partial_t U\|^{\frac{6}{5}} \|\nabla U\|^{\frac{6}{5}} \,\mathrm{dx}\right]^{\frac{5}{3}} \leq$$

$$\leq \frac{\nu}{16}\|\partial_t \nabla U\|^2_{L^2} + c\left[\left(\int_\Omega \|\partial_t U\|^2 \,\mathrm{dx}\right)^{\frac{3}{5}} \left(\int_\Omega \|\nabla U\|^3 \,\mathrm{dx}\right)^{\frac{2}{5}}\right]^{\frac{5}{3}} \leq$$

$$\leq \frac{\nu}{16}\|\partial_t \nabla U\|^2_{L^2} + c\|\partial_t U\|^2_{L^2}\|\nabla U\|^2_{L^3} \tag{4.13}$$

while the absolute value of the second, which is of lower order, can be estimated in the same way[1].

Finally if we sum equation (4.6) with equation (4.8), consider estimates (4.7), (4.9), (4.10), (4.11) and (4.13) and take into account (4.12) we get

$$\frac{\partial}{\partial t}\left[\|\partial_t U\|^2_{L^2} + \nu\|\nabla U\|^2_{L^2} + \frac{C}{2+2\mu}\|\nabla U\|^{2+2\mu}_{L^{2+2\mu}}\right] +$$

$$+\|\partial_t U\|^2_{L^2} + \nu\|\partial_t \nabla U\|^2_{L^2} + 2(C-\delta)\int_\Omega \|\partial_t \nabla U\|^2 \|\nabla U\|^{2\mu} \,\mathrm{dx} +$$

$$+4C\mu \int_\Omega (\nabla U \cdot \partial_t \nabla U)^2 \|\nabla U\|^{2\mu-2} \,\mathrm{dx} \leq$$

$$\leq c\left[\|\nabla U\|^{2+2\mu}_{L^{2+2\mu}} + \|\partial_t U\|_{L^2}\|\nabla U\|^2_{L^3} + \|f\|^2_{L^2} + \|\partial_t f\|^2_{L^2} + 1\right], \tag{4.14}$$

where $\epsilon$ is a positive constant which can be as small as we need and $\delta$ is

$$\delta = \begin{cases} \epsilon C + \dfrac{\lambda^2}{\gamma} & \text{when } \mu = 0.5 \\ \epsilon C + \left(\dfrac{\lambda^2}{\gamma}\right)^{2\mu} \dfrac{1}{2\mu}\left[\dfrac{4\mu-2}{\nu\mu}\right]^{2\mu-1} & \text{when } \mu > 0.5. \end{cases} \tag{4.15}$$

Now we want to use Gronwall lemma 1 to obtain the thesis. However, due to the presence of the first term of right side of (4.14) we have to check that $\|\nabla U\|_{L^2(L^3)}$ is bounded. To do it, we start from first a priori estimate theorem (14) which shows that $\|\nabla U\|_{L^{2+2\mu}(L^{2+2\mu})}$ is bounded and use the following inequality

$$\|\nabla U\|^2_{L^2(L^3)} \leq \|\nabla U\|^2_{L^{2+2\mu}(L^3)} T^{\frac{\mu}{\mu+1}} \leq c(|\Omega|) T^{\frac{\mu}{\mu+1}} \|\nabla U\|^2_{L^{2+2\mu}(L^{2+2\mu})}.$$

□

[1] It can be estimated also in a better way, but it is worthless here.

## 4.3 Global Existence Theorem

In this section we are going to show that under compatibility conditions on initial and boundary data, if the initial values of velocity and external forces are regular enough, solution of EVSF model (4.1) exists in $H^1(0,T;L^2(\Omega))\cap L^{2+2\mu}(0,T;W^{1,2+2\mu}_{0,\text{div}}(\Omega))$ for every $T>0$.

**Theorem 16 (Existence)** *If condition (4.5) on $C$ is satisfied, if initial datum $U_0 \in W^{1,2+2\mu}_{0,\text{div}}(\Omega)$ and if $f \in L^2(0,T;L^2(\Omega))$ and $\partial_t f \in L^2(0,T;L^2(\Omega))$, then a generalized solution of EVSF problem exists in $H^1(0,T;L^2(\Omega))\cap L^{2+2\mu}(0,T;W^{1,2+2\mu}_{0,\text{div}}(\Omega))$.*

**Proof:** Let us define

$$J^{1,1}_{2+2\mu,2} = H^1(0,T;L^2(\Omega))\cap L^{2+2\mu}(0,T;W^{1,2+2\mu}_{0,\text{div}}(\Omega))$$

and let us take $a^l$, an $L^2(\Omega)$-orthonormal base of $W^{1,2+2\mu}_{0,\text{div}}(\Omega)$ with the non-restrictive hypothesis that $a^1 = U_0$. Now we introduce

$$V^n = \sum_{l=1}^{n} c_{ln}(t)a^l(x),$$

where the coefficients $c_{ln}$ are chosen to satisfy the differential equation

$$\int_\Omega\Big(\partial_t V^n a^l + (\nu + C\|\nabla V^n\|^{2\mu})\nabla V^n\cdot\nabla a^l + V^n_j\partial_j V^n a^l\Big)\,\mathrm{dx} = \tag{4.16}$$

$$= \frac{\lambda^2}{2\gamma}\int_\Omega \partial_h V^n_j \partial_h V^n \partial_j a^l\,\mathrm{dx} + \int_\Omega f a^l\,\mathrm{dx} \qquad \forall\, l = 1,\dots,n.$$

We observe that a priori estimates of theorems 14 and 15 hold for $V^n$ too. Since this is an autonomous first order differential equation with $c_{ln}(t)$ as unknowns and since from the first a priori estimate we have that

$$\max_{t\in[0,T]}\sum_{l=1}^{n} c^2_{ln}(t) = \|V^n\|^2_{L^\infty(L^2)}$$

is bounded with respect to $n$, we can conclude that, since this differential equation is Lipschitz in the unknown $c_{ln}$ on every compact set, this is enough to assure existence and uniqueness of $c_{ln}$.

From the sequence $V^n$ we will have to choose subsequences which converge in some sense. For simplicity, these subsequences will be still denoted by $V^n$. Thus, thanks to the uniform boundedness of $\|V^n\|_{J^{1,1}_{2+2\mu,2}}$, the sequence converges to an element in $J^{1,1}_{2+2\mu,2}$ strongly in $L^2(0,T;L^2(\Omega))$ and weakly in $J^{1,1}_{2+2\mu,2}$. If we use now the imbedding theorem

$$\|V^n\|_{L^4(L^4)} \leq c(|\Omega|,T)\|\partial_t \nabla V^n\|^{\frac{3}{2}}_{L^2(L^2)}\|V^n\|^{\frac{1}{4}}_{L^2(L^2)} \tag{4.17}$$

we have that $\|V^n\|_{L^4(L^4)}$ is uniformly bounded, which together with strong convergence in $L^2(0,T;L^2(\Omega))$ assures strong convergence of $V^n$ to $V$ in the norm of $L^q(0,T;L^q(\Omega))$ for any $1 \leq q < 4$.

From equations (4.16), $V^n$ must satisfy

$$\int_0^T\!\!\int_\Omega (\partial_t V^n + V^n_j \partial_j V^n)\Phi + (\nu + C\|\nabla V^n\|^{2\mu})\nabla V^n \cdot \nabla \Phi \,\mathrm{dx}\,\mathrm{dt} =$$

$$= \frac{\lambda^2}{2\gamma}\int_0^T\!\!\int_\Omega \partial_h V^n_j \partial_h V^n \partial_j \Phi \,\mathrm{dx}\,\mathrm{dt} + \int_0^T\!\!\int_\Omega f\Phi \,\mathrm{dx}\,\mathrm{dt} \tag{4.18}$$

where $\Phi \in P^n$ is an arbitrary function obtained as a linear combination of $a^l(x)$ with coefficients $d_l(t)$, where this coefficients are absolutely continuous functions of time with square summable first derivatives. Let us try to pass to the limit in (4.18) assuming $\Phi$ to be fixed. This may be done without any problem in the first[2] and last term using the above properties of the sequence $V^n$. To pass to the limit in the strongly nonlinear second and third terms we use an idea of Minty and Browder in [29, 31, 30, 6]. We introduce the functions

$$A^k_i(\nabla V^n) = [\nu + C\|\nabla V^n\|^{2\mu}]\partial_k V^n_i - \frac{\lambda^2}{2\gamma}\sum_h \partial_h V^n_i \partial_h V^n_k$$

which, in view of the a priori estimate theorem 14, are uniformly bounded in $L^{\frac{2+2\mu}{1+2\mu}}(0,T;L^{\frac{2+2\mu}{1+2\mu}}(\Omega))$ and therefore converge weakly in this space to functions $B^k_i(x,t)$. Therefore the limiting equation of (4.18) is

$$\int_0^T\!\!\int_\Omega \Big[(\partial_t V_i + V_j \partial_j V_i)\Phi_i + B^k_i \partial_k \Phi_i \,\mathrm{dx}\Big]\,\mathrm{dt} = \int_0^T\!\!\int_\Omega f_i \Phi_i \,\mathrm{dx}\,\mathrm{dt} \tag{4.19}$$

[2]To show this for the convective term we have to use a Hölder inequality and strong convergence in $L^q$ for every $q < 4$.

which is valid for every $\Phi$ in $P^n$ for every $n$ and therefore is valid in $P = \bigcup_{n=1}^{\infty} P^n$. It is now easy to verify that this implies that it is valid for any $\Phi$ in $J^{1,1}_{2+2\mu,2}$. We now need the following lemma

**Lemma 9** *If condition on C (4.5) is satisfied, using the definition of $A_l^k$, for any two differentiable functions $v'$ and $v''$ we have*

$$r(v',v'')=\sum_{k,l}[A_l^k(\nabla v')-A_l^k(\nabla v'')](\partial_k v_l'-\partial_k v_l'')\geq\frac{\nu}{2}\sum_{k,l}(\partial_k v_l'-\partial_k v_l'')^2$$

**Proof:** We introduce $v^\tau = \tau v' + (1-\tau)v''$ and we have

$$r(v',v'')=\sum_{k,l}\left[\int_0^1\frac{d}{d\tau}A_l^k(\nabla v^\tau)\,\mathrm{d}\tau\right](\partial_k v_l'-\partial_k v_l'')=$$

$$=\sum_{i,j,k,l}\left[\int_0^1\frac{\partial A_l^k(\nabla v^\tau)}{\partial(\partial_j v_i^\tau)}\frac{\partial}{\partial\tau}(\partial_j v_i^\tau)\,\mathrm{d}\tau\right](\partial_k v_l'-\partial_k v_l'')=$$

$$=\sum_{i,j,k,l}\left[\int_0^1\frac{\partial A_l^k(\nabla v^\tau)}{\partial(\partial_j v_i^\tau)}\,\mathrm{d}\tau\right](\partial_j v_i'-\partial_j v_i'')(\partial_k v_l'-\partial_k v_l'')=$$

$$=\sum_{i,j,k,l}\int_0^1\Big[(\nu+C\|\nabla v^\tau\|^{2\mu})\delta_k^j\delta_l^i+2C\mu\|\nabla v^\tau\|^{2\mu-2}\partial_j v_i^\tau\partial_k v_l^\tau$$

$$-\frac{\lambda^2}{2\gamma}\partial_j v_k^\tau\delta_l^i-\frac{\lambda^2}{2\gamma}\partial_j v_l^\tau\delta_k^i\Big]\mathrm{d}\tau\,(\partial_j v_i'-\partial_j v_i'')(\partial_k v_l'-\partial_k v_l'') \tag{4.20}$$

We estimate now the second last term of (4.20), when $\mu > 0.5$, with

$$\frac{\lambda^2}{2\gamma}\left|\sum_{i,j,k}\partial_j v_k^\tau(\partial_j v_i'-\partial_j v_i'')(\partial_k v_i'-\partial_k v_i'')\right|\leq$$

$$\leq\frac{\lambda^2}{2\gamma}\|\nabla v^\tau\|\,\|\nabla v'-\nabla v''\|^{\frac{1}{\mu}}\|\nabla v'-\nabla v''\|^{\frac{2\mu-1}{\mu}}\leq$$

$$\leq\frac{\nu}{4}\|\nabla v'-\nabla v''\|^2+\left(\frac{\lambda^2}{2\gamma}\right)^{2\mu}\frac{1}{2\mu}\left[\frac{4\mu-2}{\mu\nu}\right]^{2\mu-1}\|\nabla v^\tau\|^{2\mu}\|\nabla v'-\nabla v''\|^2$$

and, when $\mu = 0.5$, with

$$\frac{\lambda^2}{2\gamma}\sum_{i,j,k}|\partial_j v_k^\tau|\,|\partial_j v_i'-\partial_j v_i''|\,|\partial_k v_i'-\partial_k v_i''|\leq\frac{\lambda^2}{2\gamma}\|\nabla v^\tau\|\,\|\nabla v'-\nabla v''\|^2$$

Evaluating the last term of (4.20) in the same way and substituting these estimates into (4.20) and dropping the second last term, we get the thesis.

□

Therefore we have proven that

$$\int_0^T\int_\Omega [A_l^k(\nabla V^n) - A_l^k(\nabla\eta)](\partial_k V_l^n - \partial_k\eta_l)\,\mathrm{dx}\,\mathrm{dt} \geq 0$$

and assuming now $\eta$ to be an element of $P^n$ we have

$$-\int_0^T\int_\Omega (\partial_t V_i^n + V_k^n\partial_k V_i^n - f_i)(V_i^n - \eta_i) + A_l^k(\nabla\eta)(\partial_k V_l^n - \partial_k\eta_l)\,\mathrm{dx}\,\mathrm{dt} \geq 0.$$

We now want to pass to the limit as $n \to \infty$. This does not present any problem except in the second term where it is necessary to verify that the functions $V_k^n V^n$ converge strongly in the $L^{\frac{2+2\mu}{1+2\mu}}$-norm, since $\nabla V^n$ is in $L^{2+2\mu}(0,T;L^{2+2\mu}(\Omega))$ and it converges weakly in this space. Indicating $p = (2+2\mu)/(1+2\mu)$

$$\|V_k^n V_l^n - V_k\partial_k V_l\|_{L^p(L^p)} \leq \|(V_k^n - V_k)V_l^n\|_{L^p(L^p)} + \|V_k(V_l^n - V_l)\|_{L^p(L^p)} \leq$$
$$\leq \|V_k^n - V_k\|_{L^{2p}(L^{2p})}\|V_l^n\|_{L^{2p}(L^{2p})} + \|V_k\|_{L^{2p}(L^{2p})}\|V_l^n - V_l\|_{L^{2p}(L^{2p})}$$

and, since $2p < 4$, the right side of this inequality tends to zero as $n \to \infty$. Therefore

$$-\int_0^T\int_\Omega (\partial_t V_i + V_k\partial_k V_i - f_i)(V_i - \eta_i) + A_l^k(\nabla\eta)(\partial_k V_l - \partial_k\eta_l)\,\mathrm{dx}\,\mathrm{dt} \geq 0. \quad (4.21)$$

Inequality (4.21) has been proved for a function $\eta$ from $P$ but it is also valid for $\eta \in J^{1,1}_{2+2\mu,2}$. If we add it to inequality (4.19) with $\Phi = V - \eta$ and $\eta = V - \epsilon\xi$, $\epsilon > 0$ and $\xi \in J^{1,1}_{2+2\mu,2}$, we get

$$\int_0^T\int_\Omega [B_l^k - A_l^k(\nabla V - \epsilon\nabla\xi)]\partial_k\xi_l\,\mathrm{dx}\,\mathrm{dt} \geq 0.$$

Therefore this inequality, due to the fact that

$$A_l^k(\nabla V - \epsilon\nabla\xi)\partial_k\xi_l =$$
$$= A_l^k(\nabla V)\partial_k\xi_k + [A_l^k(\nabla V - \epsilon\nabla\xi) - A_l^k(\nabla V)]\partial_k\xi_l \leq A_l^k(\nabla V)\partial_k\xi_l,$$

is valid for an arbitrary function $\xi$ and, after $\epsilon \to 0$, since $J^{1,1}_{2+2\mu,2}$ is a linear set, the equality sign holds. Therefore (4.19) coincides with (4.18) with $\Phi \in J^{1,1}_{2+2\mu,2}$

$$\int_0^T\int_\Omega (\partial_t V + V_j\partial_j V)\Phi + (\nu + C\|\nabla V\|^{2\mu})\nabla V\cdot\nabla\Phi\,\mathrm{dx}\,\mathrm{dt} =$$

$$= \frac{\lambda^2}{2\gamma}\int_0^T\int_\Omega \partial_h V_j \partial_h V \partial_j \Phi \,\mathrm{dx}\,\mathrm{dt} + \int_0^T\int_\Omega f\Phi \,\mathrm{dx}\,\mathrm{dt}$$

which is the definition of generalized solution of our problem.

□

## 4.4 Uniqueness and stability

Without further assumptions we are able to show uniqueness of our solution in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2+2\mu}(\Omega))$, stability and, with an assumption on external forces $f$, the fact that the $L^2(\Omega)$-norm of the solution tends to zero as time goes to infinity.

**Theorem 17 (Uniqueness)** *Under the same hypothesis of existence theorem on $\lambda$, problem (4.1) possesses no more than one generalized solution in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2+2\mu}(\Omega))$.*

**Proof:** Let $U = V' - V''$ be the difference of two solutions of (4.1) in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2+2\mu}(\Omega))$. It must satisfy, in a distributional sense,

$$\partial_t U - \partial_k[A^k(\nabla V') - A^k(\nabla V'')] + V_j'\partial_j V' - V_j''\partial_j V'' = 0,$$

with $U(0) = 0$. We multiply by $U$ and integrate over $\Omega$ to get

$$\partial_t\|U\|_{L^2}^2 + 2\int_\Omega [A^k(\nabla V') - A^k(\nabla V'')]\partial_k U \,\mathrm{dx} + 2\int_\Omega U_j \partial_j V' U \,\mathrm{dx} = 0$$

and, using the property of $A^k$, we arrive at

$$\partial_t\|U\|_{L^2}^2 + \nu\|\nabla U\|_{L^2}^2 \le 2\left|\int_\Omega U_j\partial_j V' U \,\mathrm{dx}\right| \le \tag{4.22}$$

$$\le 2\|U\|_{L^2}\|U\|_{L^6}\|\nabla V'\|_{L^{2+2\mu}} \le c\|U\|_{L^2}\|\nabla U\|_{L^2}\|\nabla V'\|_{L^{2+2\mu}} \le$$

$$\le c\|U\|_{L^2}^2\|\nabla V'\|_{L^{2+2\mu}}^2 + \frac{\nu}{2}\|\nabla U\|_{L^2}^2.$$

Now we can integrate (4.22) in time and using Gronwall lemma 1 we get $U = 0$.

□

**Theorem 18 (Decay)** *If $C$ satisfies (4.5), if $f$ is in $L^2(0,+\infty;L^2(\Omega))$ and is vanishing in $L^2(\Omega)$ as $t \to +\infty$, if $U_0 \in L^2(\Omega)$, then the $L^2(\Omega)$-norm of our solution of (4.1) tends to zero as time tends to infinity.*

**Proof:** We multiply (4.1) by $e^{\bar{c}t}$ and we define $W = Ue^{\bar{c}t}$. Then we multiply by $W$, integrate over $\Omega$ and use Hölder inequality

$$\frac{1}{2}\partial_t\|W\|^2_{L^2} - \bar{c}\|W\|^2_{L^2} + \nu\|\nabla W\|^2_{L^2} + C\int_\Omega\|\nabla U\|^{2\mu}\|\nabla W\|^2\,\mathrm{dx} \leq \tag{4.23}$$

$$\leq \|fe^{\bar{c}t}\|_{L^2}\|W\|_{L^2} + \frac{\lambda^2}{2\gamma}\int_\Omega\|\nabla U\|\,\|\nabla W\|^2\,\mathrm{dx}.$$

When $\mu = 0.5$ this side of inequality becomes

$$\leq \frac{c}{\epsilon}\|f\|^2_{L^2}e^{2\bar{c}t} + \epsilon\|\nabla W\|^2_{L^2} + \frac{\lambda^2}{2\gamma}\int_\Omega\|\nabla U\|^{2\mu}\|\nabla W\|^2\,\mathrm{dx},$$

while when $\mu > 0.5$

$$\leq \frac{c}{\epsilon}\|f\|^2_{L^2}e^{2\bar{c}t} + \epsilon\|\nabla W\|^2_{L^2} +$$

$$+\left(\frac{\lambda^2}{2\gamma}\right)^{2\mu_1}\frac{1}{2\mu}\left(\frac{2\mu-1}{\mu\nu}\right)^{2\mu-1}\int_\Omega\|\nabla U\|^{2\mu}\|\nabla W\|^2\,\mathrm{dx} + \frac{\nu}{2}\|\nabla W\|^2_{L^2}$$

and, if we take $\bar{c}$ and $\epsilon$ small enough, we get

$$\partial_t\|W\|^2_{L^2} \leq c\|f\|^2_{L^2}e^{2\bar{c}t}. \tag{4.24}$$

Since the $L^2$-norm of $f$ goes to zero with time, for each $\bar{\epsilon} > 0$ there is a $\tau_1$ such that for every $t > \tau_1$ we have $\|f\|_{L^2} \leq \bar{\epsilon}$.

Integrating (4.24) in time we get

$$\|W(t)\|^2_{L^2} \leq \|W(t_0)\|^2_{L^2} + c\int_{t_0}^t\|f(s)\|^2_{L^2}e^{2\bar{c}s}\,\mathrm{ds}$$

$$\|U(t)\|^2_{L^2} \leq e^{-2\bar{c}t}\left[\|U(t_0)\|^2_{L^2}e^{2\bar{c}t_0} + c\int_{t_0}^t\|f(s)\|^2_{L^2}e^{2\bar{c}s}\,\mathrm{ds}\right] \leq$$

$$\leq \|U(t_0)\|^2_{L^2}e^{-2\bar{c}(t-t_0)} + c\int_{t_0}^t\|f(s)\|^2_{L^2}e^{-2\bar{c}(t-s)}\,\mathrm{ds}.$$

Now we take $t_0$ such that $U(t_0) \in L^2(\Omega)$ and such that $t_0 > \tau_1$ and we obtain

$$\|U(t)\|_{L^2}^2 \leq \|U(t_0)\|_{L^2}^2 e^{-2\bar{c}(t-t_0)} + c\bar{\epsilon}e^{-2\bar{c}t}\frac{1}{2\bar{c}}\Big[e^{2\bar{c}s}\Big]_{s=t_0}^{s=t} =$$

$$= \|U(t_0)\|_{L^2}^2 e^{-2\bar{c}(t-t_0)} + \frac{c\bar{\epsilon}}{2\bar{c}}\Big(1 - e^{-2\bar{c}(t-t_0)}\Big).$$

Now if $t$ is large enough, $t \geq \tau_2$, we have

$$\|U(t_0)\|_{L^2}^2 e^{-2\bar{c}(t-t_0)} \leq \epsilon \qquad \forall\, t \geq \tau_2$$

and for $t \geq \max(\tau_1, \tau_2)$ we proved that

$$\|U(t)\|_{L^2}^2 \leq c\bar{\epsilon}.$$

□

**Corollary 4 (Exponential decay)** *If $f = 0$ we easily have*

$$\|U(t)\|_{L^2} \leq \|U_0\|_{L^2} e^{-\bar{c}t}.$$

**Theorem 19 (Stability)** *Under the same hypothesis on $\lambda$ of existence theorem, two solutions $V'$ and $V''$ in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2+2\mu}(\Omega))$ with different initial data $V_0'$, $V_0''$ and different external forces $f'$ and $f''$ satisfy*

$$\|V' - V''\|_{L^\infty(L^2)} \leq \|V_0' - V_0''\|_{L^2} \exp c(V_0', V_0'', f', f'', \Omega, T).$$

**Proof:** We call $f = f' - f''$ and $U = V' - V''$. In exactly the same way as uniqueness theorem 17 we get

$$\partial_t \|U\|_{L^2}^2 + 2\nu\|\nabla U\|_{L^2}^2 \leq c\|U\|_{L^2}^2 \|\nabla V'\|_{L^{2+2\mu}}^2 + \frac{c}{\epsilon}\|f\|_{L^2}^2 + 2\epsilon\|\nabla U\|_{L^2}^2$$

which implies

$$\|U\|_{L^2} \leq \|U_0\|_{L^2} \exp\Big[c\|f\|_{L^1(L^2)} + c\|U\|_{L^\infty(L^2)}^2 \|\nabla V'\|_{L^2(L^{2+2\mu})}^2\Big]$$

and, thanks to estimate of $V'$ and $V''$ using initial data, we get the thesis.

□

**Corollary 5** *If* $\|\nabla V'\|_{L^\infty(0,+\infty;L^2)} \leq \alpha\nu$ *for* $\alpha$ *small enough, if* $(f' - f'')$ *is in* $L^1(0, +\infty; L^2(\Omega))$ *and is vanishing in* $L^2(\Omega)$ *as* $t \to \infty$*, then the difference* $V' - V''$ *decays to zero as time goes to infinity.*

**Proof:** Estimating the convective term as

$$\left| \int_\Omega U_k \partial_k V_i' U_i \, \mathrm{dx} \right| \leq c \|\nabla V'\|_{L^2} \|U\|_{L^2}^{\frac{1}{2}} \|\nabla U\|_{L^2}^{\frac{3}{2}} \leq \bar{c} \|\nabla V'\|_{L^2} \|\nabla U\|_{L^2}^2$$

and taking $\alpha < \frac{1}{\bar{c}}$, we have

$$\|U\|_{L^2} \partial_t \|U\|_{L^2} + c\|\nabla U\|_{L^2}^2 \leq \|f\|_{L^2} \|U\|_{L^2}.$$

Therefore

$$\|V' - V''\|_{L^2} \leq \|V_0' - V_0''\|_{L^2} \mathrm{e}^{-ct} + \int_0^t \|f\|_{L^2} \mathrm{e}^{-c(t-\tau)} \, \mathrm{d}\tau$$

and the thesis follows.

□

# Chapter 5

# Numerical solutions

In this chapter we are going to show some numerical results concerning the previously presented models in a two-dimensional square cavity. We are going to use a finite-element method to spatially discretize our equations and a three-step splitting scheme for time advancing with divergence free restriction taken into account in the first and last step.

We looked for numerical resolution at various Reynolds numbers, ranging from $10^2$ to $10^6$ and using Navier-Stokes equations, SF, EV and EVSF large eddy simulation models.

## 5.1 The numerical scheme

We show here only the numerical scheme for EVSF large eddy simulation model. To have the numerical scheme for the other three sets of differential equations, it is enough to neglect the appropriate terms. We just remind EVSF large eddy simulation model (4.1) in its two-dimensional with parameter $\mu = 0.5$ version

$$\begin{cases} \partial_t U_i + \sum_{j=1}^{2} U_j \partial_j U_i = -\frac{1}{\rho}\partial_i P + \sum_{j=1}^{2} \partial_j (\nu + C\|\nabla U\|^{\frac{1}{2}})\partial_j U_i + \\ \qquad\qquad - \sum_{j,l=1}^{2} \partial_j \left[ \frac{\lambda^2}{2\gamma} \partial_l U_i \partial_l U_j \right] + f_i \qquad i = 1,2 \\ \sum_{j=1}^{2} \partial_j U_j = 0,. \end{cases} \tag{5.1}$$

$$U_{|\partial\Omega} = U_{\partial\Omega} \qquad U_{|t=0} = U_0 \qquad \sum_{j=1}^{2} \partial_j U_{0j} = 0 \qquad U_{0|\partial\Omega} = U_{\partial\Omega|t=0},$$

where $\gamma$ is usually taken equal to 6, $\lambda$ is usually taken equal to the grid size and $C$ is usually the square of the grid size.

We start by introducing the three-step splitting scheme proposed by Glowinsky [18] and analyzed by Klouček and Rys [21], where our problem is splitted into three steps. In the first and the last one we take the viscous and eddy viscosity terms $\alpha$-implicit and $\beta$-explicit ($\beta = 1-\alpha$, $0<\alpha<1$), convective and space-filtering terms are taken explicit and pressure term fully implicit. Divergence free restriction is taken also into account in both steps. In step two the viscous and eddy-viscosity terms are taken $\beta$-implicit and $\alpha$-explicit, convective and space-filtering terms are taken fully implicit and pressure term explicit. Step two is not divergence free. Introducing discrete spaces $V_{hg}$ and $Q_h$

$$V_{hg} = \{v_h \in (H^1_g(\Omega))^2 | v_{h|K} \in (\mathbb{P}^2)^2 \forall K \in \tau_h\},$$

$$Q_h = \{v_h \in L^2_0(\Omega) | v_{h|K} \in \mathbb{P}^1 \forall K \in \tau_h\},$$

where $\tau_h$ is a triangulation of $\Omega$, $L^2_0$ is the space of $L^2$ functions with a vanishing average over $\Omega$ (this last choice is to remove the degree of freedom in the definition of the pressure), the scalar product $(\ ,\ )$ of $L^2(\Omega)$, $\theta \in (0, 0.5)$, $\theta_b = 1-2\theta$ and $\Delta t$ is the time step, we have to solve a linear Stokes problem, a nonlinear Burgers problem and another linear Stokes problem. They are:

*First step*: Find $u^{n+\theta} \in V_{hg}$ and $p^{n+\theta} \in Q_h$ such that $\forall \phi \in V_{h0}$ and $\forall \psi \in Q_h$

$$\frac{1}{\theta\Delta t}(u_i^{n+\theta}, \phi_i) + \alpha\Big(\big[\nu + C\|\nabla u^n\|\big]\nabla u_i^{n+\theta}, \nabla\phi_i\Big) - (p^{n+\theta}, \partial_i\phi_i) =$$

$$= \frac{1}{\theta\Delta t}(u_i^n, \phi_i) - \beta\Big(\big[\nu + C\|\nabla u^n\|\big]\nabla u_i^n, \nabla\phi_i\Big) - (u_j^n\partial_j u_i^n, \phi_i) +$$

$$+\frac{\lambda^2}{2\gamma}(\partial_l u_i^n \partial_l u_j^n, \partial_j\phi_i) + (f_i^{n+\theta}, \phi_i);$$

$$(\partial_i u_i^{n+\theta}, \psi) = 0;$$

*Second step*: Find $u^{n+1-\theta} \in V_{hg}$ such that $\forall \phi \in V_{h0}$

$$\frac{1}{\theta_b\Delta t}(u_i^{n+1-\theta}, \phi_i) + \beta\Big(\big[\nu + C\|\nabla u^n\|\big]\nabla u_i^{n+1-\theta}, \nabla\phi_i\Big) +$$

$$+(u_j^{n+1-\theta}\partial_j u_i^{n+1-\theta}, \phi_i) - \frac{\lambda^2}{2\gamma}(\partial_l u_i^{n+1-\theta}\partial_l u_j^{n+1-\theta}, \partial_j \phi_i) =$$

$$= \frac{1}{\theta_b \Delta t}(u_i^{n+\theta}, \phi_i) - \alpha\Big(\big[\nu + C\|\nabla u^n\|\big]\nabla u_i^{n+\theta}, \nabla\phi_i\Big)$$

$$+(p^{n+\theta}, \partial_i\phi_i) + (f_i^{n+1-\theta}, \phi_i); \qquad (5.2)$$

*Third step*: Find $u^{n+1} \in V_{hg}$ and $p^{n+1} \in Q_h$ such that $\forall\phi \in V_{h0}$ and $\forall\psi \in Q_h$

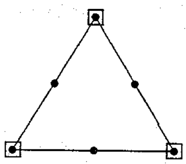
$$\frac{1}{\theta\Delta t}(u_i^{n+1}, \phi_i) + \alpha\Big(\big[\nu + C\|\nabla u^n\|\big]\nabla u_i^{n+1}, \nabla\phi_i\Big) - (p^{n+1}, \partial_i\phi_i) =$$

$$= \frac{1}{\theta\Delta t}(u_i^{n+1-\theta}, \phi_i) - \beta\Big(\big[\nu + C\|\nabla u^n\|\big]\nabla u_i^{n+1-\theta}, \nabla\phi_i\Big) +$$

$$-(u_j^{n+1-\theta}\partial_j u_i^{n+1-\theta}, \phi_i) + \frac{\lambda^2}{2\gamma}(\partial_l u_i^{n+1-\theta}\partial_l u_j^{n+1-\theta}, \partial_j\phi_i) + (f_i^{n+1}, \phi_i);$$

$$(\partial_i u_i^{n+1}, \psi) = 0.$$

We choose triangular elements with velocity nodes on vertices and in the middle of each side and with pressure nodes on vertices. Velocity components are approximated with parabolic piecewise approximation, while pressure with linear piecewise approximation.



- velocity point

□ pressure point

This choice of Taylor-Hood elements satisfies the compatibility condition of Ladyzhenskaya-Babuška-Brezzi (LBB) to avoid spurious modes (see Quarteroni and Valli [38, sec. 9.2.2])

$$\mathrm{Ker} B^T = 0,$$

where $B_{mn} = (\mathrm{div}\,\phi^n, \psi^m)$ is the $M_h \times 2N_h$-matrix of $L^2$ scalar product of divergence of velocity test functions and pressure test functions, where $N_h$ is the number of velocity test functions degrees of freedom and $M_h$ the number of pressure test functions degrees of freedom. Having satisfied LBB we do not need to use any penalty method to solve our two Stokes problems.

LES discretized model is now reduced to two symmetric linear Stokes systems of $2N_h + M_h$ equations in $2N_h + M_h$ unknowns and a nonlinear Burgers system of $2N_h$ equations in $2N_h$ unknowns with positive definite and symmetric linear part.

We could solve linear systems using Gauss method, since, if time step is constant and if we do not use eddy viscosity, we have to do only one triangular factorization (which has complexity $O(n^3)$ operations) for every grid we use and then at every time step we just have to solve two triangular linear systems (with a complexity of $O(n^2)$ operations). However we choose a conjugate gradient method which converges, in our case, very rapidly.

The Stokes problem can be expressed in matricial notation with

$$\begin{cases} Au + B^T p = f \\ Bu = 0, \end{cases} \tag{5.3}$$

where $A$ is symmetric and positive defined and thus its inverse exists and is positive defined. We therefore obtain pressure $p$ solving

$$BA^{-1}B^T p = BA^{-1} f$$

with conjugate gradient method since $BA^{-1}B^T$ is symmetric and positive defined. We note here that during each iteration we have to solve an additional linear system in order to invert $A^{-1}$; we solve it using a conjugate gradient method. Finally we get velocity $u$ solving

$$Au = -B^T p + f.$$

We solve nonlinear Burgers system using a modified version of conjugate gradient method (MCG) with Polack-Riebere strategy to minimize cost functional

$$J(v) = \frac{1}{2}\sqrt{< A^{-1}F(v), F(v) >},$$

where $A$ is the matrix representing the linear part of (5.2) and $F$ the is the nonlinear part. Details can be found in Girault-Raviart [17]. Defining $F$ as the right part of (5.2), MCG is

*MCG step one*: Given $u^m \in V_{hg}$, find $z^m \in V_{h0}$ such that

$$\frac{1}{\theta_b \Delta t}(z_i^m, \phi_i) + \beta\Big(\big[\nu + C\|\nabla u^n\|\big]\nabla z_i^m, \nabla\phi_i\Big) = \frac{1}{\theta_b \Delta t}(u_i^m, \phi_i) +$$

$$+\beta\Big(\big[\nu + C\|\nabla u^n\|\big]\nabla u_i^m, \nabla\phi_i\Big) + (u_j^m\partial_j u_i^m, \phi_i) +$$
$$-\frac{\lambda^2}{2\gamma}(\partial_l u_i^m \partial_l u_j^m, \partial_j\phi_i) - (F_i, \phi_i);$$

*MCG step two*: Find $g^m \in V_{h0}$ such that

$$\frac{1}{\theta_b\Delta t}(g_i^m, \phi_i) + \beta\Big(\big[\nu + C\|\nabla u^n\|\big]\nabla g_i^m, \nabla\phi_i\Big) =$$
$$= \frac{1}{\theta_b\Delta t}(z_i^m, \phi_i) + \beta\Big(\big[\nu + C\|\nabla u^n\|\big]\nabla z_i^m, \nabla\phi_i\Big) + (u_j^m\partial_j\phi_i + \phi_j\partial_j u_i^m, z_i^m) +$$
$$-\frac{\lambda^2}{2\gamma}(\partial_l u_i^m \partial_l\phi_j + \partial_l\phi_i\partial_l u_j^m, \partial_j z_i^m);$$

*Polack-Riebere variant*: Evaluate

$$\sigma^m = \frac{(\nabla(g_i^m - g_i^{m-1}), \nabla g_i^m)}{(\nabla g_i^{m-1}, \nabla g_i^{m-1})}, \qquad \forall m > 1$$
$$\bar{g}^m = g^m + \sigma^m\bar{g}^{m-1} \qquad \forall m > 1;$$

*MCG step three*: Evaluate first terms of the cost functional to be minimized

$$J(u^m) = \frac{1}{2}\Big[\beta\Big(\big[\nu + C\|\nabla u^n\|\big]\partial_j z_i^m, \partial_j z_i^m\Big) + \frac{1}{\theta_b\Delta t}(z_i^m, z_i^m)\Big],$$
$$DJ(u^m) = \frac{1}{2}\Big[\beta\Big(\big[\nu + C\|\nabla u^n\|\big]\partial_j\bar{g}_i^m, \partial_j\bar{g}_i^m\Big) + \frac{1}{\theta_b\Delta t}(\bar{g}_i^m, \bar{g}_i^m)\Big];$$

*MCG step four*: Find $v^m \in V_{h0}$ such that

$$\frac{1}{\theta_b\Delta t}(v_i^m, \phi_i) + \beta\Big(\big[\nu + C\|\nabla u^n\|\big]\nabla v_i^m, \nabla\phi_i\Big) =$$
$$= \frac{1}{\theta_b\Delta t}(\bar{g}_i^m, \phi_i) + \beta\Big(\big[\nu + C\|\nabla u^n\|\big]\nabla\bar{g}_i^m, \nabla\phi_i\Big) +$$
$$+(u_j^m\partial_j\bar{g}_i^m + \bar{g}_j^m\partial_j u_i^m, \phi_i) - \frac{\lambda^2}{2\gamma}(\partial_l u_i^m\partial_l\bar{g}_j^m + \partial_l\bar{g}_i^m\partial_l u_j^m, \partial_j\phi_i);$$

*MCG step five*: Define

$$[t_i^m, \cdot] = (2\bar{g}_j^m\partial_j\bar{g}_i^m, \cdot) - \frac{\lambda^2}{2\gamma}(2\partial_l\bar{g}_i^m\partial_l\bar{g}_j^m, \partial_j\,\cdot\,),$$

$$D^2J(u^m) = \beta\Big(\big[\nu + C\|\nabla u^n\|\big]\partial_j v_i^m, \partial_j v_i^m\Big) + \frac{1}{\theta_b\Delta t}(v_i^m, v_i^m) + [t_i^m, z_i^m],$$

$$D^3J(u^m) = 3[t_i^m, v_i^m];$$

*MCG step six*: Find $w^m \in V_{h0}$ such that

$$\frac{1}{\theta_b\Delta t}(w_i^m, \phi_i) + \beta\Big(\big[\nu + C\|\nabla u^n\|\big]\nabla w_i^m, \nabla\phi_i\Big) = [t_i^m, \phi_i];$$

*MCG step seven*: Evaluate the cost functional of $u^m - \rho g^m$, which is a fourth order polynomial in $\rho$ and is therefore equal to its Taylor fourth-order expansion

$$D^4J(u^m) = \beta\Big(\big[\nu + C\|\nabla u^n\|\big]\partial_j w_i^m, \partial_j w_i^m\Big) + \frac{1}{\theta_b\Delta t}(w_i^m, w_i^m) + (z_i^m, t_i^m),$$

$$J(u^m - \rho\bar{g}^m) = J(u^m) - DJ(u^m)\rho + D^2J(u^m)\frac{\rho^2}{2} +$$

$$-D^3J(u^m)\frac{\rho^3}{6} + D^4J(u^m)\frac{\rho^4}{24};$$

*MCG step eight*: Find, the positive root $\rho^m$ of

$$\frac{dJ}{d\rho}(u^m - \rho^m\bar{g}^m) = 0;$$

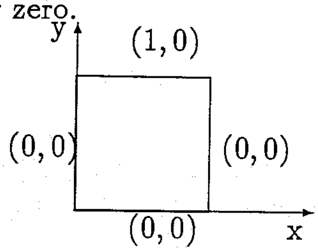
*MCG step nine*: Evaluate new approximation of solution

$$u^{m+1} = u^m - \rho^m\bar{g}^m.$$

As initial condition the value of velocity coming from previous Stokes step is taken, the algebraic equation of step eight is solved using simply the unweighted Newton method, while the four linear systems at each iteration can be solved using Gauss method for band-matrices, since, if $\Delta t$, $\theta$ and $\alpha$ are constant in time and if eddy viscosity is not used, they are associated to a matrix which remains the same at every iteration of MCG method and at every time step. However we experimented that conjugate gradient in our case is much faster and less memory consuming.

# 5.2 Test problem

Our test problem is the so called square-cavity. We take as domain $\Omega$ a square of side one, therefore its boundary $\partial\Omega$ are the four sides. We take as starting velocity zero and as boundary velocity zero on three sides and $(1,0)$ on the fourth side, which represents either a moving wall or an uniform flow independent from our solution. This leads to a discontinuity in boundary velocity as the first component of velocity on the corners of the fourth side can be one or zero.



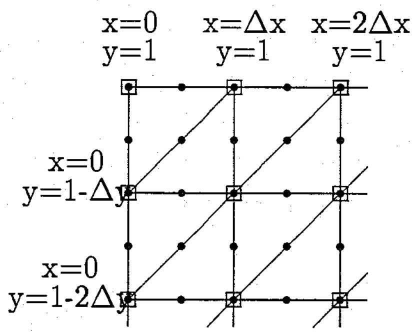

To overcome this problem we can take the first component of velocity on the fourth side as a concave parabola but this did not give any advantage in our numerical experiments.

Another irregularity is represented by the starting velocity which is discontinuous on the whole fourth side. This could be solved by starting with a velocity different from zero, but from the numerical point of view, this discontinuity does not cause serious problems since we will be more interested in the stationary solution.

We usually take as final time the time when stationary solution is reached using a stopping test which requires that the value of velocity in every point does not change more than $10^{-3}$ times the maximum value of that velocity component. We use as a comparison tool horizontal velocity profiles on the vertical line in the middle of the square and vertical velocity profiles on the horizontal line in the middle of the square. In this way we can compare our results with the results obtained by Ghia [16] for Navier-Stokes problem using a very thin mesh up to Reynolds number 10,000. For larger Reynolds numbers we will instead present a graphical representation of the velocity field using arrows and the streamfunction sign (negative values in gray and positive ones in white), which show very well the vortices. The streamfunction is a scalar function $\Psi$ obtained solving

$$\begin{cases} -\Delta\Psi = \partial_2 U_1 - \partial_1 U_2 \qquad \forall\, x \in \Omega \\ \Psi = 0 \qquad \forall\, x \in \partial\Omega. \end{cases}$$

We use a triangulation with 800 right-angled triangles:




Time step $\Delta t$ is taken, unless otherwise noted, equal to 0.05. $\gamma$ is equal to 6, $\lambda$ is equal to the average of $\Delta x$ and $\Delta y$ while $C$ is $C_0\lambda^2$, where the value of $C_0$ depends on the simulation.

The boundary condition for horizontal velocity on point $(x, y) = (0, 1)$ and on point $(x, y) = (1, 1)$ can be chosen equal to zero or to one. If we state that velocity be zero on these two points, we are automatically stating that horizontal velocity between points $(0, 1)$ and $(\Delta x, 1)$ and between points $(1-\Delta x, 1)$ and $(1, 1)$ be different from one. If we state that horizontal velocity on points $(0, 1)$ and $(1, 1)$ be one, we are automatically stating that horizontal velocity between points $(0, 1-\Delta y)$ and $(0, 1)$ and between $(1, 1-\Delta y)$ and $(1, 1)$ be different from zero. In our numerical simulations we will experiment both conditions and show that our solution depends strongly on this condition.

Our program has been tested on several a priori-known analytical solution to check for errors and to show its capability to reach stationary solutions.

## 5.3 Results for low Reynolds numbers

At low Reynolds numbers we simulated the results of Navier-Stokes equations and SF turbulence model since it is worthless to add an eddy viscosity term when velocity gradients are small.

We start our simulations examining the result of our test problem in the $Re = 100$ laminar case up to stationary time 8.7 seconds. Here Navier-Stokes equations and SF turbulence model give exactly the same results, with a very slightly better[1] result for Navier-Stokes equations. However, the most critical difference lies in the temporal step $\Delta t$ which was taken equal to $10^{-1}$ seconds for Navier-Stokes equations and $10^{-3}$ seconds for SF turbulence model. This is due to the fact that our nine-steps method for Burgers' problem does not converge when the strong nonlinear SF term is present if $\Delta t$ is not small enough.

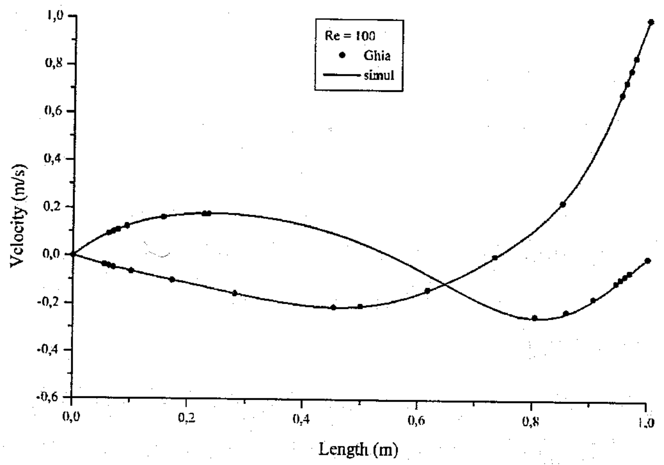



Figure 5.1: Velocity across symmetry of cavity in the $Re = 100$ NS case.

In the $Re = 400$ case, Navier-Stokes equations give also good result at stationary time 38.8 seconds with a time step of $2 \cdot 10^{-2}$ seconds. However, even reducing the time step to $10^{-4}$ seconds is not enough to let SF turbulence model predict our flow after the critical time of 0.8 seconds.

In the $Re = 1,000$ case Navier-Stokes equations reach the stationary solution after 36 seconds of simulation with a time step of $10^{-2}$ seconds. The result is in good agreement with Ghia, however there are slight discrepancies when velocity reach its maximum and minimum values. SF turbulence model is not able to work anymore at this Reynolds number and therefore we are not able to confirm the claim of Cantekin and Westerink [7] that SF model works in good agreement with Ghia up to $Re = 3,200$, at least with our algorithm. For low Reynolds numbers SF model works, but in this case the pure NS system still gives good results and it is simpler and has a faster algorithm than SF.

[1] In our experiments we take Ghia's results as the true solution and therefore *better* means *more similar to Ghia's calculations.*

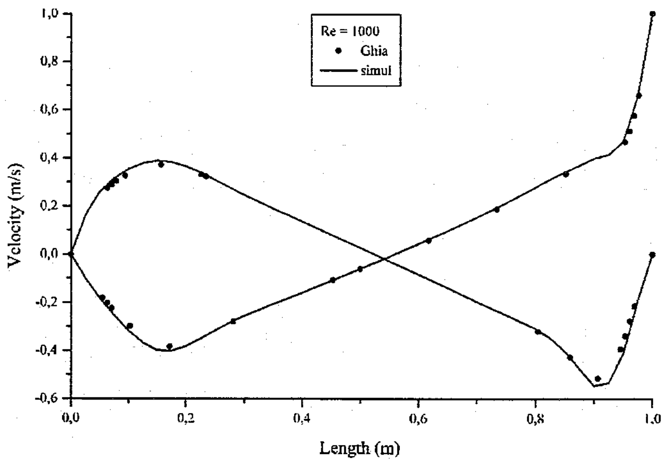



Figure 5.2: Velocity across symmetry of cavity in the $Re = 1,000$ NS case.

### 5.3.1 Comments

Our experiments show that Navier-Stokes, as it was already well-known, can be solved with good results for small Reynolds numbers. However, SF turbulence model which, according to Cantekin and Westerink [7], should lead to better results, did not prove worth the effort. In order to deal with the highly nonlinear turbulence term we had to drastically reduce the time step even in cases when our solution of Navier-Stokes equations is in good agreement with Ghia's data. This is due to the fact that the term added by SF model does not have any good numerical nor analytical properties as we showed in Chapter 3, where it does not contribute to any solution estimate. However the poor results obtained with SF model can instead be attributed to the complicated time advancement scheme which works very well for nonlinear convective terms, but has never been tested with stronger nonlinear terms.

## 5.4 Results for moderate Reynolds numbers

At moderate Reynolds numbers we simulate the transition between laminar and turbulent flow. At this range of numbers SF turbulence model does not work anymore and therefore we compare the results of Navier-Stokes equations, EV and EVSF turbulence models. From now on we will always use a time step of $10^{-2}$.

In the $Re = 3,200$ simulation of Navier-Stokes equations continues to work, but gives very inaccurate results. However, if we change the boundary condition on the upper angles from horizontal zero velocity to horizontal velocity equal to one, we obtain a much more reliable result at stationary time of 209 seconds, even through absolute velocity is underestimated. If we use EV model, we obtain slightly better results at stationary time of 117 seconds with the latter boundary condition and with $C = C_0\Delta x \cdot \Delta y = 10^{-2}\Delta x \cdot \Delta y$. If we use instead $C_0 = 10^{-3}$ we obtain again underestimated absolute velocity which probably is due to the same reason as the previous analogous result.

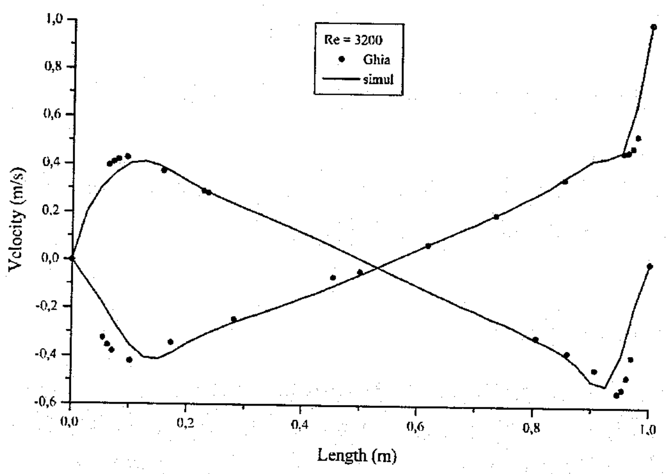



Figure 5.3: Velocity across symmetry of cavity in the $Re = 3,200$ EV case with $C_0 = 10^{-2}$.

In the $Re = 5,000$ case Navier-Stokes equations with the former boundary condition diverge after 169 seconds, while with the latter boundary condition they produce a slightly unstable result (it does not reach stationary state after 266 seconds of simulation) but in quite good agreement with Ghia's data. If we introduce an eddy viscosity with coefficient $C_0 = 10^{-3}$ we obtain the same result. If we try to use the EVSF model, in order to avoid divergence after some seconds of simulation, we have to take $C_0 = 10^{-1}$, which is still much less that what is suggested by existence theorem in Chapter 3 and is exactly what is suggested in most LES literature. With this coefficient we reach stationary state at time 44 but the results, due to the larger eddy viscosity coefficient $C_0$, are not in good agreement with Ghia's data.

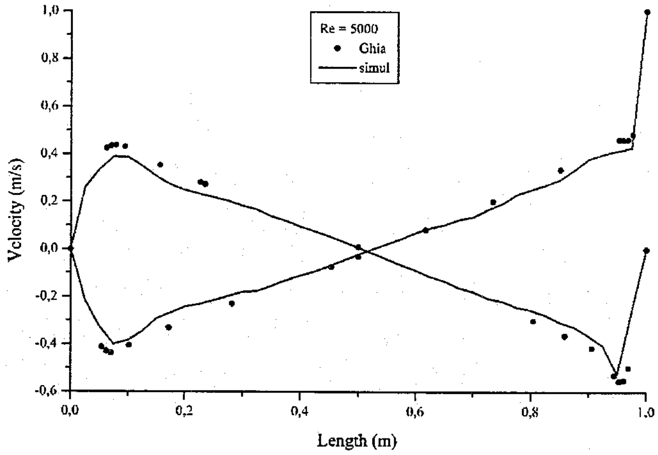



Figure 5.4: Velocity across symmetry of cavity in the $Re = 5,000$ NS case.

If we put zero boundary condition on upper angles, we always obtain overestimated absolute value of velocity.

### 5.4.1 Comments

Our experiments on moderate Reynolds numbers show that Navier-Stokes equations are still quite good at predicting fluid velocity even through they start to feel the effects of a slight difference in boundary conditions. EV models here work more or less like Navier-Stokes equations and therefore are suggested only to improve convergence or stability, but with coefficient $C_0$ in the range between $10^{-3}$ and $10^{-2}$ to avoid having a completely different solution as we had when we tried to use the $10^{-1}$ value suggested in most LES literature. EVSF model suffers the same weaknesses as SF model: it needs a large viscosity (a large eddy viscosity in this case) to work and it does not give better results.

## 5.5 Results for high Reynolds numbers

In the $Re = 7,500$ case we obtain an EV model's result in good agreement with Ghia's data when we take $C_0 = 10^{-2}$ even if velocity are still a bit irregular after 278 seconds of simulation.

In the $Re = 10,000$ case we obtain results in poor agreement with Ghia.

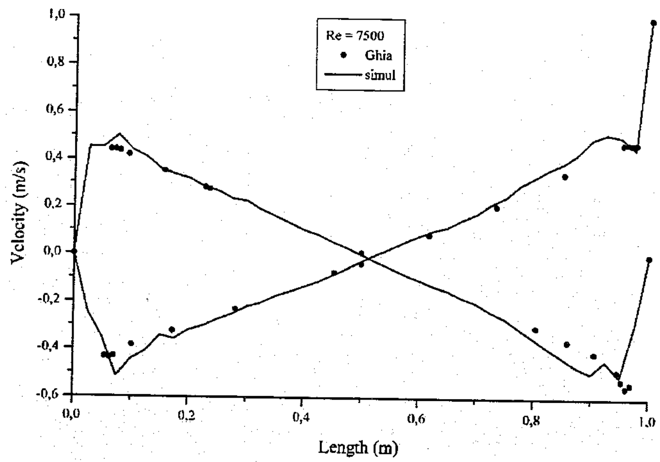



Figure 5.5: Velocity across symmetry of cavity in the $Re$ = 7,500 EV case with $C_0 = 10^{-2}$.

The best simulation is obtained with an EV model with $C_0 = 5 \cdot 10^{-2}$ at stationary time of 63 seconds. The two vortices in the bottom of the cavity join while this is not prescribed by Ghia's data and we have no trace of the vortex in the upper left corner. However we prescribe correctly the birth of another vortex in the lower right corner. If we try other EV coefficient we obtain similar, but slightly worse, results.

In the $Re = 100,000$ case the eddy viscosity coefficient $C_0 = 10^{-2}$ does not lead to a stationary solution any more. Therefore we work with $C_0 = 2 \cdot 10^{-2}$ (which reaches a quasi steady-state after 200 seconds of simulation) and $C_0 = 5 \cdot 10^{-2}$ (convergence after 66 seconds). The two vortices in the lower corners have joined in the same horizontally stretched vortex, while another vortex has developed in the lower right corner. In the former case we have also another vortex in the lower left corner, but in this case velocities are quite irregular.

The $Re = 1,000,000$ case is almost identical to the $Re = 100,000$ case. It is worth noting that in both these cases the eddy which should appear at $Re = 3,200$ and should be fully developed at such large Reynolds numbers has only a slight appearance in our $C_0 = 2 \cdot 10^{-2}$ case. This fact can be attributed to the boundary condition on the upper left corner and on the quite large horizontal space step we used.

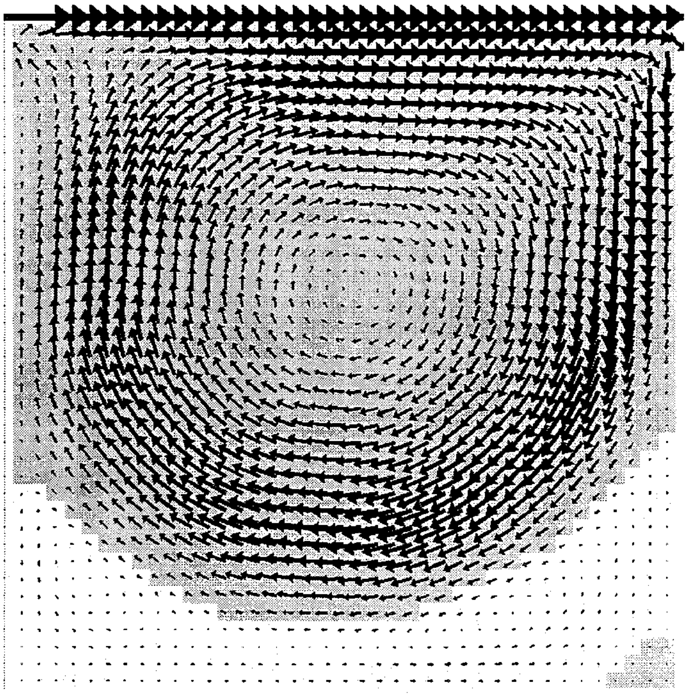


Figure 5.6: Cavity in the $Re = 10,000$ EV case with $C_0 = 5 \cdot 10^{-2}$.

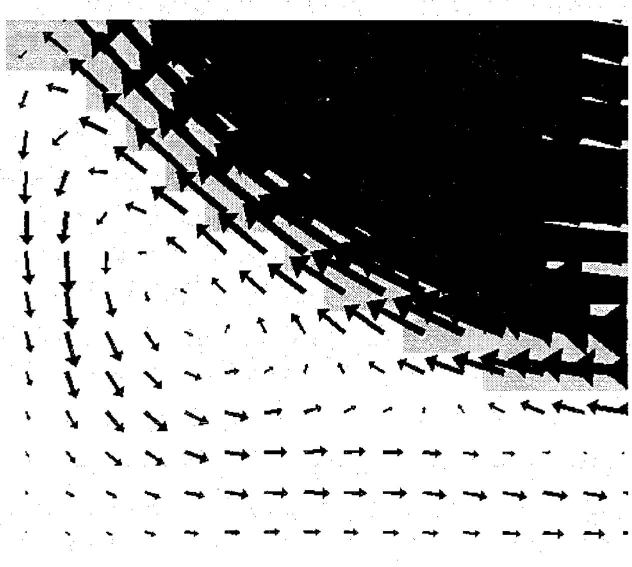


Figure 5.7: Left lower quarter of cavity in the $Re = 10,000$ EV case with $C_0 = 5 \cdot 10^{-2}$.

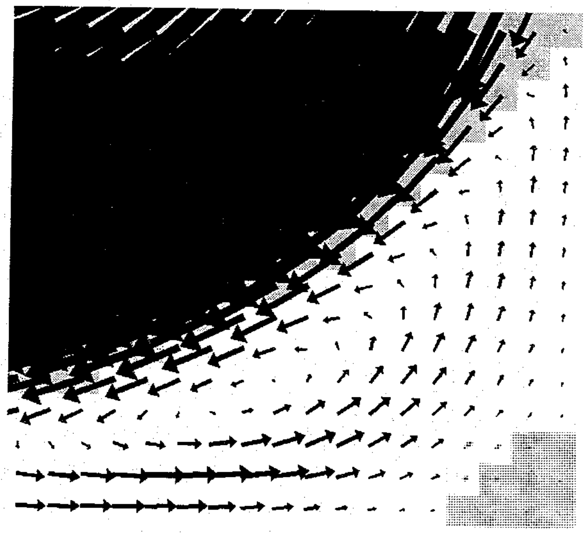


Figure 5.8: Right lower quarter of cavity in the $Re = 10,000$ EV case with $C_0 = 5 \cdot 10^{-2}$.

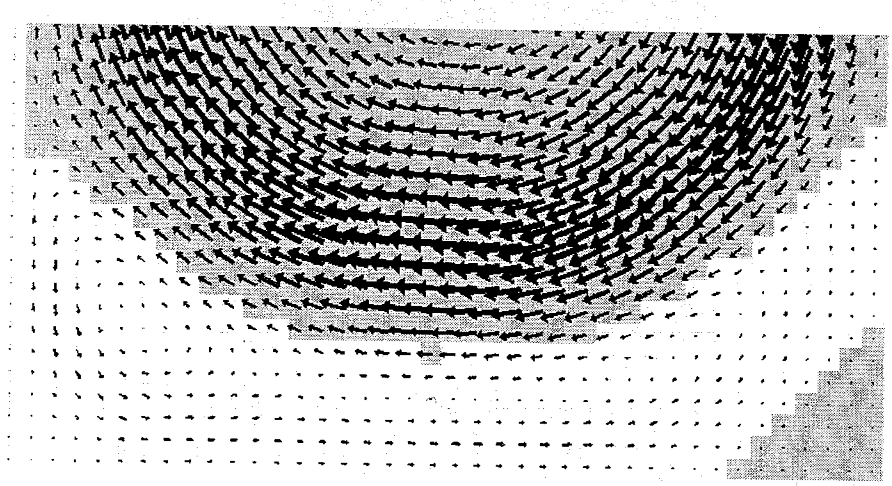


Figure 5.9: Lower half of cavity in the $Re = 100,000$ EV case with $C_0 = 5 \cdot 10^{-2}$.

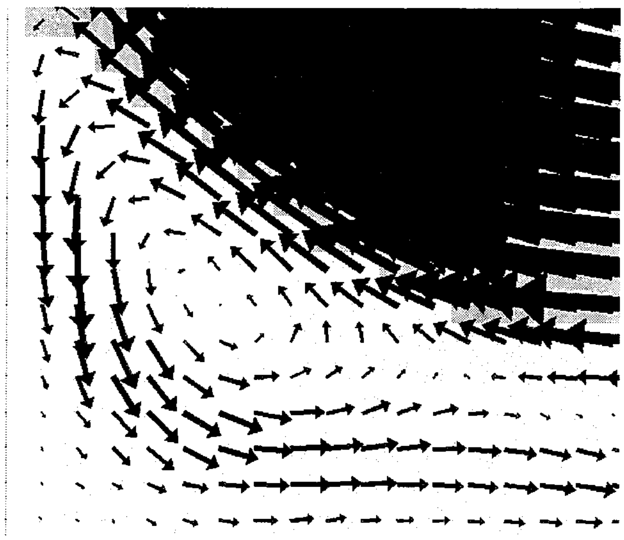


Figure 5.10: Left lower quarter of cavity in the $Re = 100,000$ EV case with $C_0 = 5 \cdot 10^{-2}$.

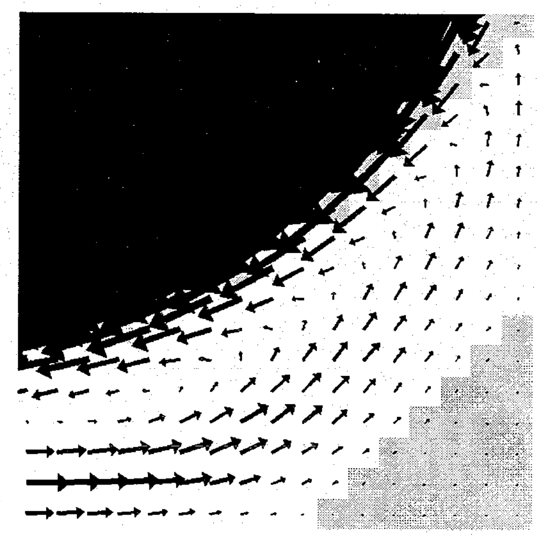


Figure 5.11: Right lower quarter of cavity in the $Re = 100,000$ EV case with $C_0 = 5 \cdot 10^{-2}$.

### 5.5.1 Comments

Our experiments on high Reynolds numbers show that EV models here work quite well, at least up to $Re = 10,000$ where we can keep the value of eddy viscosity coefficient low enough. In any case, EV models work with quite small eddy viscosity coefficient compared to the ones suggested by most LES literature (usually $10^{-1}$ or larger). Unfortunately we cannot compare the results we obtained, since Ghia's results are no more available for comparison for Reynolds number larger than $10,000$, but they are quite believable since they show the growth of the two corner vortices and the birth of another vortex. There are almost no differences between the $Re = 100,000$ and the $Re = 1,000,000$ case.

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